Mathematical Models of Systems

We use quantitative mathematical models of physical systems to design and analyze control systems. The dynamic behavior is generally described by ordinary differential equations. We will consider a wide range of systems, including mechanical, hydraulic, and electrical. Since most physical systems are nonlinear, we will discuss linearization approximations, which allow us to use Laplace transform methods.

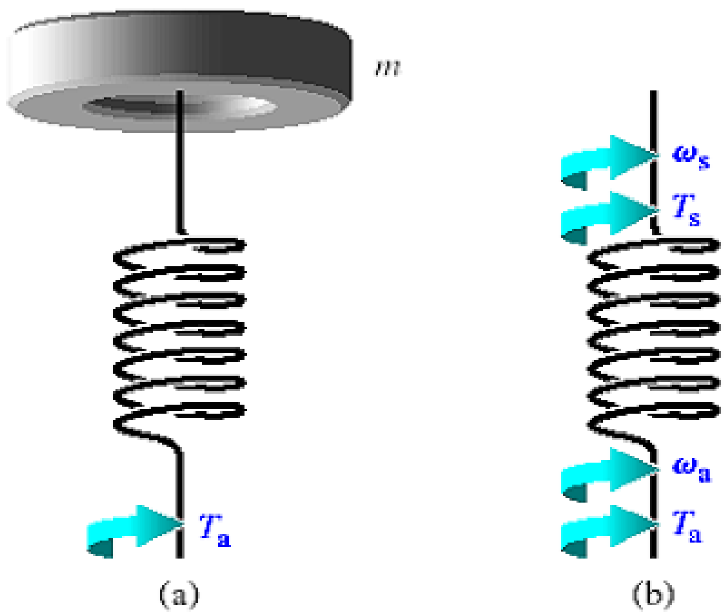
We will then proceed to obtain the input–output relationship for components and subsystems in the form of transfer functions. The transfer function blocks can be organized into block diagrams or signal-flow graphs to graphically depict the interconnections. Block diagrams (and signal-flow graphs) are very convenient and natural tools for designing and analyzing complicated control systems

Introduction

Six Step Approach to Dynamic System Problems

- Define the system and its components
- Formulate the mathematical model and list the necessary assumptions
- Write the differential equations describing the model
- Solve the equations for the desired output variables
- Examine the solutions and the assumptions
- If necessary, reanalyze or redesign the system

Differential Equation of Physical Systems



(a) Torsional spring-mass system.
(b) Spring element.

$T_a(t) - T_s(t) = 0$

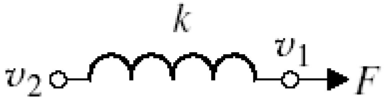
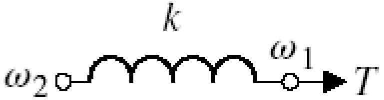
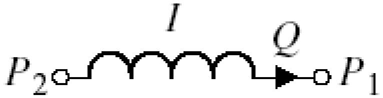
$T_a(t) = T_s(t)$

$\omega(t) = \omega_s(t) - \omega_a(t)$

$T_a(t)$ = through - variable

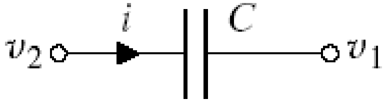
angular rate difference = across-variable

Differential Equation of Physical Systems

	Describing Equation	Energy or Power
Electrical Inductance 	$v_{21} = L \cdot \frac{d}{dt} i$	$E = \frac{1}{2} \cdot L \cdot i^2$
Translational Spring 	$v_{21} = \frac{1}{k} \cdot \frac{d}{dt} F$	$E = \frac{1}{2} \cdot \frac{F^2}{k}$
Rotational Spring 	$\omega_{21} = \frac{1}{k} \cdot \frac{d}{dt} T$	$E = \frac{1}{2} \cdot \frac{T^2}{k}$
Fluid Inertia 	$P_{21} = I \cdot \frac{d}{dt} Q$	$E = \frac{1}{2} \cdot I \cdot Q^2$

Differential Equation of Physical Systems

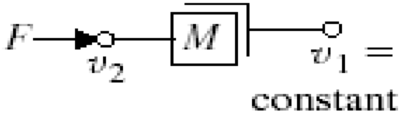
Electrical Capacitance



$$i = C \cdot \frac{d}{dt} v_{21}$$

$$E = \frac{1}{2} \cdot M \cdot {v_{21}}^2$$

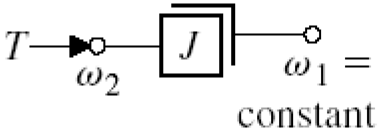
Translational Mass



$$F = M \cdot \frac{d}{dt} v_2$$

$$E = \frac{1}{2} \cdot M \cdot {v_2}^2$$

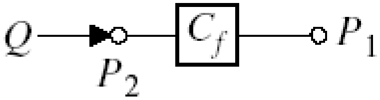
Rotational Mass



$$T = J \cdot \frac{d}{dt} \omega_2$$

$$E = \frac{1}{2} \cdot J \cdot {\omega_2}^2$$

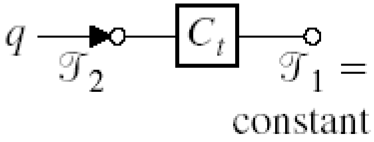
Fluid Capacitance



$$Q = C_f \cdot \frac{d}{dt} P_{21}$$

$$E = \frac{1}{2} \cdot C_f \cdot {P_{21}}^2$$

Thermal Capacitance

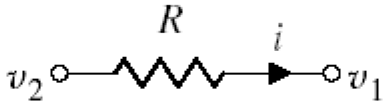


$$q = C_t \cdot \frac{d}{dt} T_2$$

$$E = C_t \cdot T_2$$

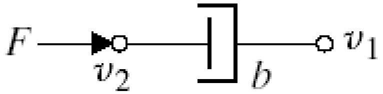
Differential Equation of Physical Systems

Electrical Resistance



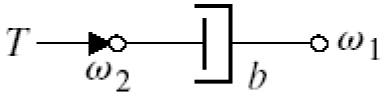
$$i = \frac{1}{R} \cdot v_{21} \qquad P = \frac{1}{R} \cdot {v_{21}}^2$$

Translational Damper



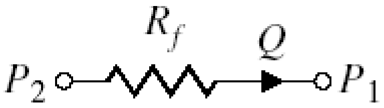
$$F = b \cdot v_{21} \qquad P = b \cdot {v_{21}}^2$$

Rotational Damper



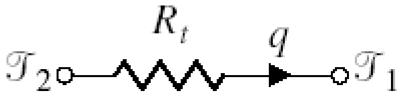
$$T = b \cdot \omega_{21} \qquad P = b \cdot {\omega_{21}}^2$$

Fluid Resistance



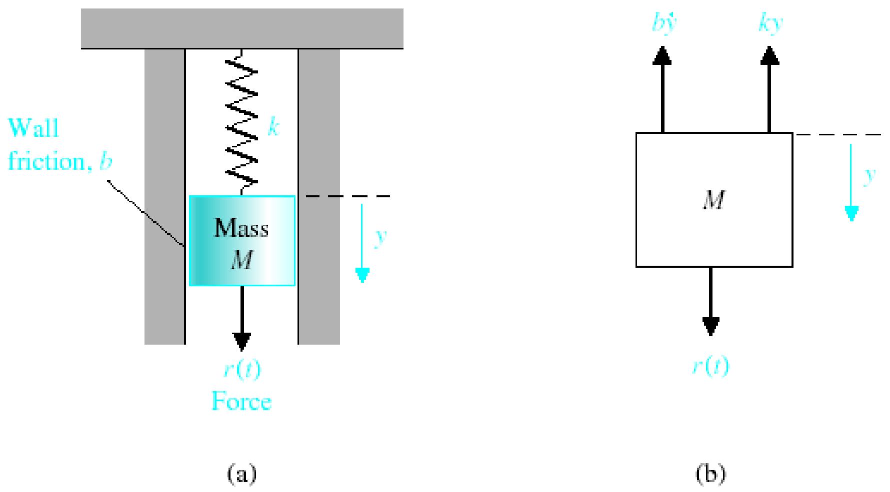
$$Q = \frac{1}{R_f} \cdot P_{21} \qquad P = \frac{1}{R_f} \cdot {P_{21}}^2$$

Thermal Resistance



$$q = \frac{1}{R_t} \cdot T_{21} \qquad P = \frac{1}{R_t} \cdot T_{21}$$

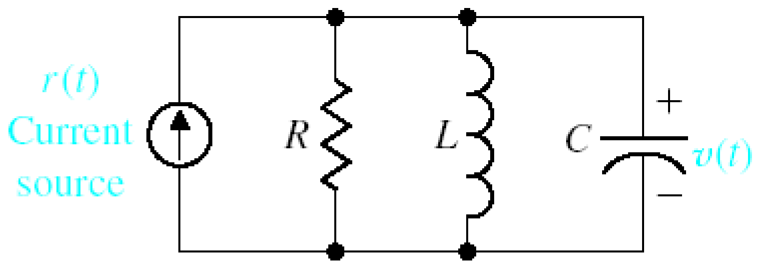
Differential Equation of Physical Systems



(a) Spring-mass-damper system.
(b) Free-body diagram.

$$M\cdot\frac{d^2}{dt^2}y(t) + b\cdot\frac{d}{dt}y(t) + k\cdot y(t) = r(t)$$

Differential Equation of Physical Systems

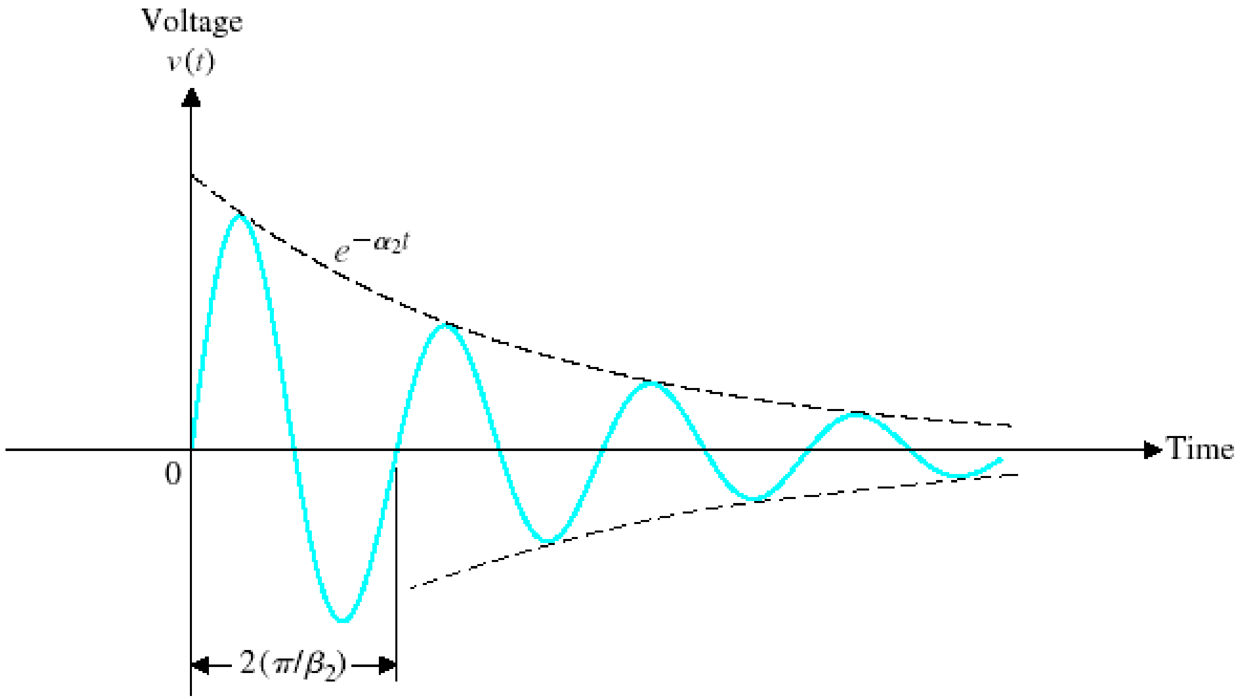


RLC circuit.

$$\frac{v(t)}{R} + C \cdot \frac{d}{dt} v(t) + \frac{1}{L} \cdot \int_0^t v(t)\, dt = r(t)$$

$$y(t) = K_1 \cdot e^{-\alpha_1 \cdot t} \cdot \sin\left(\beta_1 \cdot t + \theta_1\right)$$

Differential Equation of Physical Systems



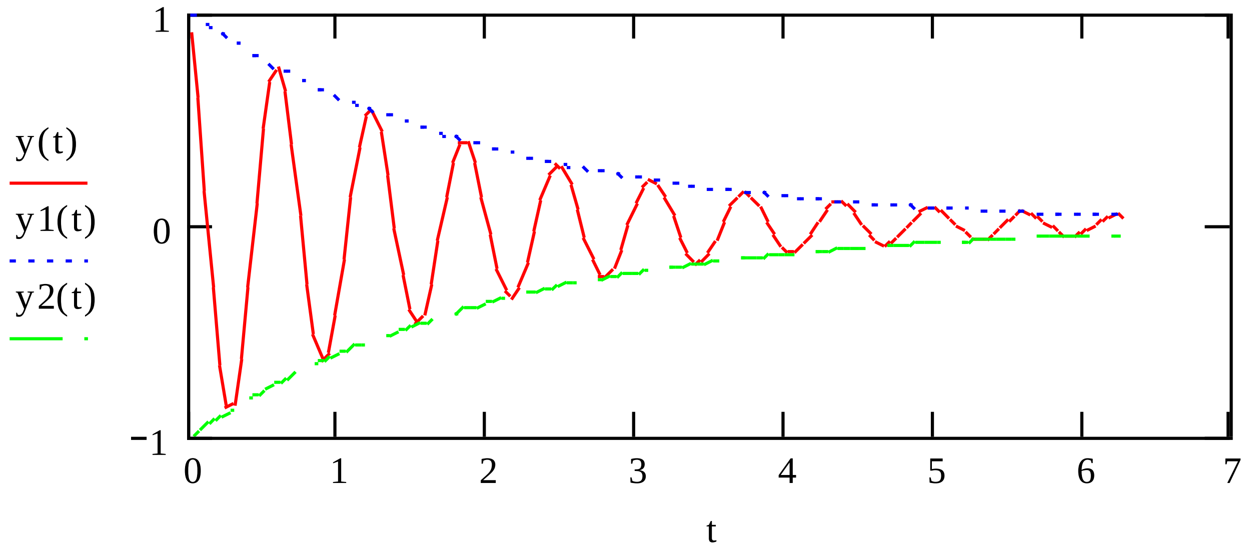
Typical voltage response for underdamped RLC circuit.

Differential Equation of Physical Systems

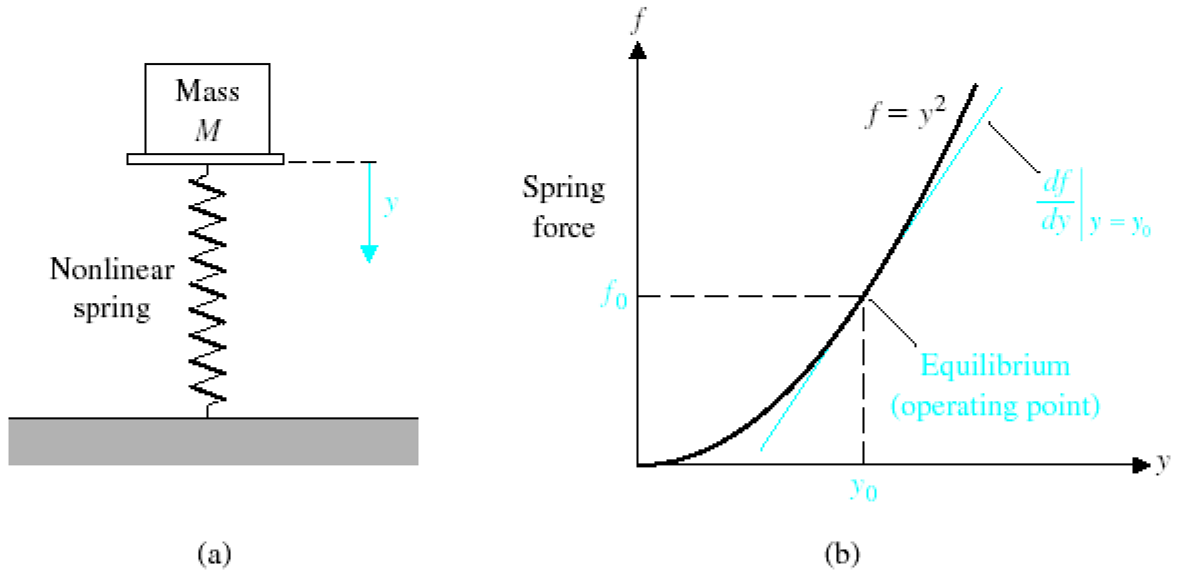
$K_2 := 1 \qquad \alpha_2 := .5 \qquad \beta_2 := 10 \qquad \theta_2 := 2$

$y(t) := K_2 \cdot e^{-\alpha_2 \cdot t} \cdot \sin\left(\beta_2 \cdot t + \theta_2\right)$

$y1(t) := K_2 \cdot e^{-\alpha_2 \cdot t} \qquad y2(t) := -K_2 \cdot e^{-\alpha_2 \cdot t}$



Linear Approximations



(a) A mass sitting on a nonlinear spring.
(b) The spring force versus y.

Linear Approximations

Linear Systems - Necessary condition

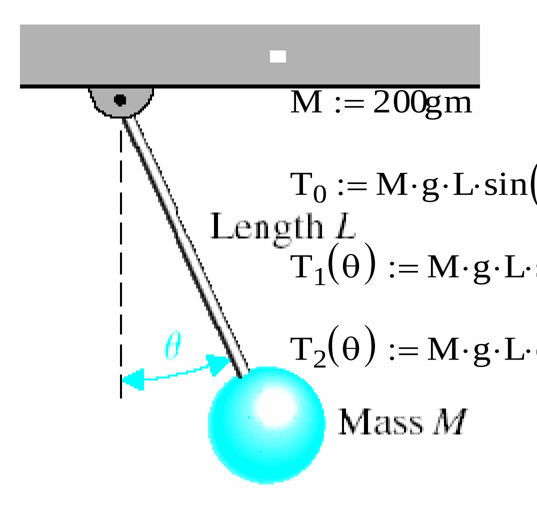
Principle of Superposition

Property of Homogeneity

Taylor Series

http://www.maths.abdn.ac.uk/%7Eigc/tch/ma2001/notes/node46.html

Linear Approximations – Example 2.1



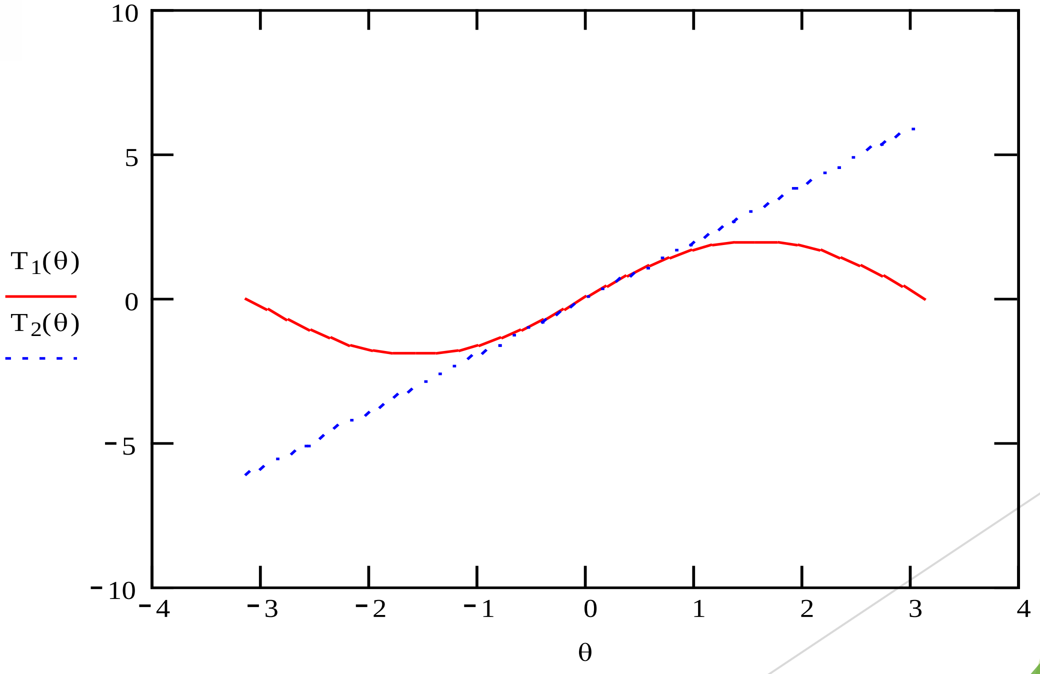
Pendulum oscillator.

$M := 200gm \qquad g := 9.8\frac{m}{s^2} \qquad L := 100cm \qquad \theta_0 := 0rad \qquad \theta := -\pi, \frac{-15\pi}{16} .. \pi$

$T_0 := M \cdot g \cdot L \cdot \sin(\theta_0)$

$T_1(\theta) := M \cdot g \cdot L \cdot \sin(\theta)$

$T_2(\theta) := M \cdot g \cdot L \cdot \cos(\theta_0) \cdot (\theta - \theta_0) + T_0$



Students are encouraged to investigate linear approximation accuracy for different valu θ_0

The Laplace Transform

Historical Perspective - Heaviside's Operators

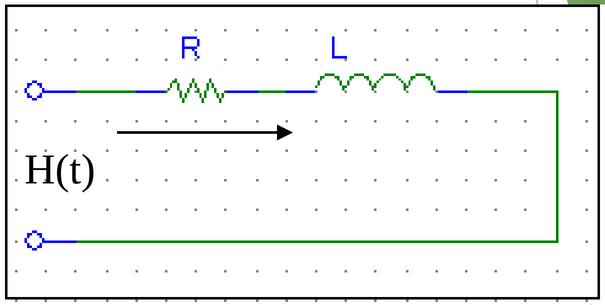
Origin of Operational Calculus (1887)

Historical Perspective - Heaviside's Operators
Origin of Operational Calculus (1887)

$$p = \frac{d}{dt}$$

$$\frac{1}{p} = \int_0^t 1\,du$$

$v =$ H(t)



Expanded in a power series

$$i = \frac{v}{Z(p)}$$

$$Z(p) = R + L\cdot p$$

$$i = \frac{1}{R + L\cdot p}\cdot H(t) = \frac{1}{L\cdot p\left(1 + \frac{R}{L\cdot p}\right)}\cdot H(t) = \frac{1}{R}\cdot\left[\frac{R}{L}\cdot\frac{1}{p} - \left(\frac{R}{L}\right)^2\cdot\frac{1}{p^2} + \left(\frac{R}{L}\right)^3\cdot\frac{1}{p^3}\ .....\right]\cdot H(t)$$

$$\frac{1}{p^n}\cdot H(t) = \frac{t^n}{n!}$$

$$i = \frac{1}{R}\cdot\left[\frac{R}{L}\cdot t - \left[\left(\frac{R}{L}\right)^2\cdot\frac{t^2}{2!} + \left(\frac{R}{L}\right)^3\cdot\frac{t^3}{3!} - ..\right]\right]$$

$$i = \frac{1}{R}\cdot\left[1 - e^{-\left(\frac{R}{L}\right)\cdot t}\right]$$

(*) Oliver Heaviside: Sage in Solitude, Paul J. Nahin, IEEE Press 1987.

The Laplace Transform

Definition

$$L(f(t)) = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt \qquad = F(s)$$

Here the complex frequency is $\qquad s = \rho + j \cdot w$

The Laplace Transform exists when

$$\int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt < \infty$$

this means that the integral converg

The Laplace Transform

Determine the Laplace transform for the functions

a) $f_1(t) := 1$ for $t \geq 0$

$$F_1(s) := \int_0^{\infty} e^{-s \cdot t}\, dt \quad = \quad -\frac{1}{s} \cdot e^{-(s \cdot t)} \qquad \frac{1}{s}$$

b) $f_2(t) = e^{-(a \cdot t)}$

$$F_2(s) = \int_0^{\infty} e^{-(a \cdot t)} \cdot e^{-(s \cdot t)}\, dt \quad = \quad -\frac{1}{s+1} \cdot e^{-[(s+a) \cdot t]} \qquad F_2(s) = \frac{1}{s+a}$$

The Laplace Transform

Evaluate the laplace transform of the derivative of a function

$$L\left(\frac{d}{dt}f(t)\right) = \int_0^{\infty} \frac{d}{dt}f(t)\cdot e^{-(s\cdot t)}\,dt$$

by the use of $\int u\,dv = u\cdot v - \int v\,du$

where $u = e^{-(s\cdot t)}$ $\quad dv = df(t)$

and, from which

$du = -s\cdot e^{-(s\cdot t)}\cdot dt$ and $v = f(t)$

we obtain

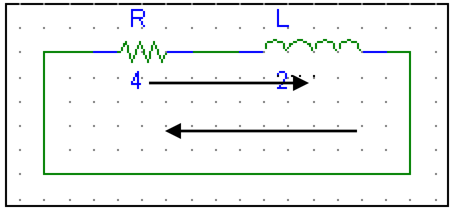
$$\int_0^{\infty} u\,dv = f(t)\cdot e^{-(s\cdot t)} - \int_0^{\infty} f(t)\cdot\left[-s\cdot e^{-(s\cdot t)}\right]dt$$

$$= -f(0+) + s\cdot\int_0^{\infty} f(t)\cdot e^{-(s\cdot t)}\,dt$$

→ $L\cdot\left(\frac{d}{dt}f(t)\right) = sF(s) - f(0+)$ note that the initial condition is included in the transform

The Laplace Transform

Practical Example - Consider the circuit.



The KVL equation is

$$4 \cdot i(t) + 2 \cdot \frac{d}{dt} i(t) = 0$$

assume i(0+) = 5 A

Applying the Laplace Transform, we have

$$\int_0^{\infty} \left(4 \cdot i(t) + 2 \cdot \frac{d}{dt} i(t) \right) \cdot e^{-(s \cdot t)} \, dt = 0 \qquad 4 \cdot \int_0^{\infty} i(t) \cdot e^{-(s \cdot t)} \, dt + 2 \cdot \int_0^{\infty} \frac{d}{dt} i(t) \cdot e^{-(s \cdot t)} \, dt = 0$$

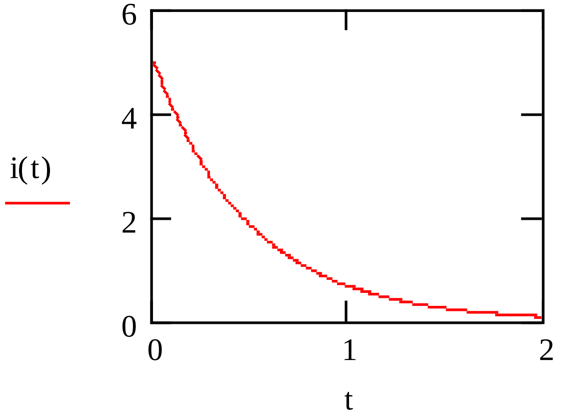
$$4 \cdot I(s) + 2 \cdot (s \cdot I(s) - i(0)) = 0 \qquad 4 \cdot I(s) + 2 \cdot s \cdot I(s) - 10 = 0$$

$$I(s) := \frac{5}{s + 2}$$

transforming back to the time domain, with our present knowledge of Laplace transform, we may say that

$$t \equiv (0, 0.01 .. 2)$$

$$i(t) \equiv 5 \cdot e^{-(2 \cdot t)}$$



The Laplace Transform

The Partial-Fraction Expansion (or Heaviside expansion theorem)

Suppose that

$$F(s) = \frac{s + z1}{(s + p1)\cdot(s + p2)}$$

or

$$F(s) = \frac{K1}{s + p1} + \frac{K2}{s + p2}$$

The partial fraction expansion indicates that F(s) consists of a sum of terms, each of which is a factor of the denominator. The values of K1 and K2 are determined by combining the individual fractions by means of the lowest common denominator and comparing the resultant numerator coefficients with those of the coefficients of the numerator before separation in different terms.

Evaluation of Ki in the manner just described requires the simultaneous solution of n equations. An alternative method is to multiply both sides of the equation by (s + pi) then setting s= - pi, the right-hand side is zero except for Ki so that

$$Ki = \frac{(s + pi)\cdot(s + z1)}{(s + p1) + (s + p2)}$$

s = - pi

The Laplace Transform

Property	Time Domain	Frequency Doma
1. Time delay	$f(t - T)\cdot u(t - T)$	$e^{-(s\cdot T)}\cdot F(s)$
2. Time scaling	$f(at)$	$\frac{1}{a}\cdot F\left(\frac{s}{a}\right)$
3. Frequency differentiation	$t\cdot f(t)$	$-\frac{d}{ds}F(s)$
4. Frequency shifting	$f(t)\cdot e^{-(a\cdot t)}$	$F(s + a)$
5. Frequency Integration	$\frac{f(t)}{t}$	$\int_0^{\infty} F(s)\,ds$
6. Initial-value Theorem	$Lim(f(t)) = f(0)$ t -> 0	$Lim(s\cdot F(s))$ s -> infinite
7. Final-value Theorem	$Lim(f(t))$ t -> infinite	$Lim(s\cdot F(s))$ s -> 0

The Laplace Transform

Useful Transform Pairs

The Laplace Transform

Consider the mass-spring-damper system

$$Y(s) = \frac{(Ms + b)\cdot yo}{Ms^2 + bs + k} \qquad \text{equation 2.2}$$

$$y(s) = \frac{\left(s + \frac{b}{M}\right)\cdot\left(y_o\right)}{\left[s^2 + \left(\frac{b}{M}\right)\cdot s + \frac{k}{M}\right]} = \frac{\left(s + 2\cdot\zeta\cdot\omega_n\right)}{s^2 + 2\cdot\zeta\cdot\omega_n + \omega_n^{\,2}}$$

$$s1 = -\left(\zeta\cdot\omega_n\right) + \omega_n\cdot\sqrt{\zeta^2 - 1}$$

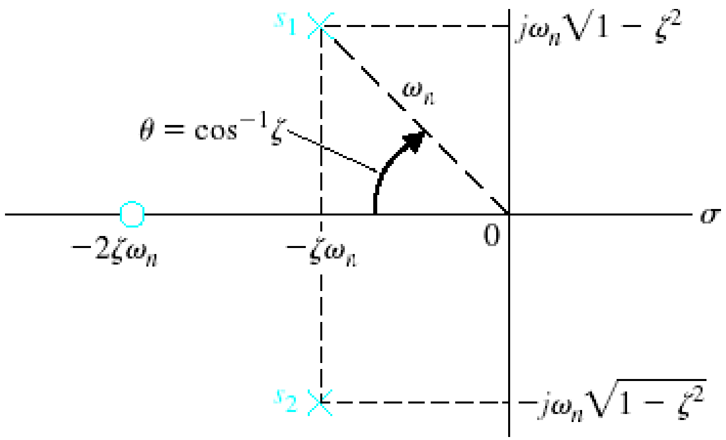
$$\omega_n = \sqrt{\frac{k}{M}} \qquad \zeta = \frac{b}{\left(2\cdot\sqrt{k\cdot M}\right)}$$

$$s2 = -\left(\zeta\cdot\omega_n\right) - \omega_n\cdot\sqrt{\zeta^2 - 1}$$

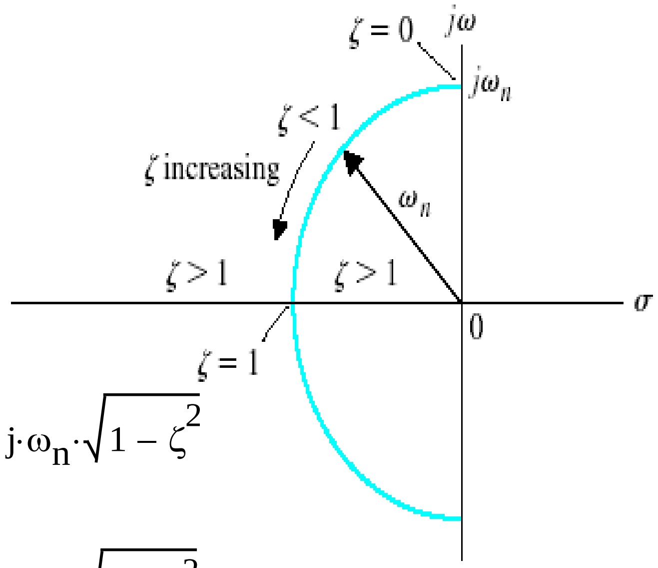
Roots

Real
Real repeated
Imaginary (conjugates)
Complex (conjugates)

An s-plane plot of the poles and zeros of Y(s).



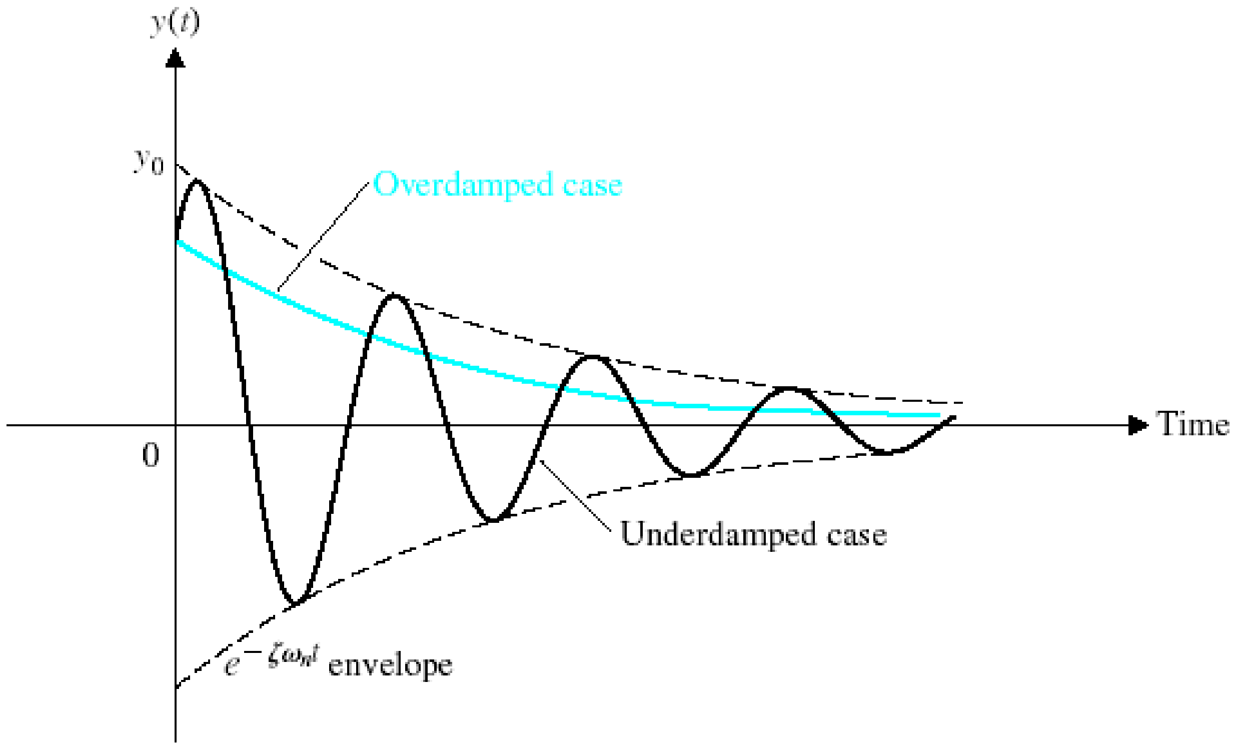
The locus of roots as z varies with vn constant.



$$s1 = -\left(\zeta\cdot\omega_n\right) + j\cdot\omega_n\cdot\sqrt{1 - \zeta^2}$$

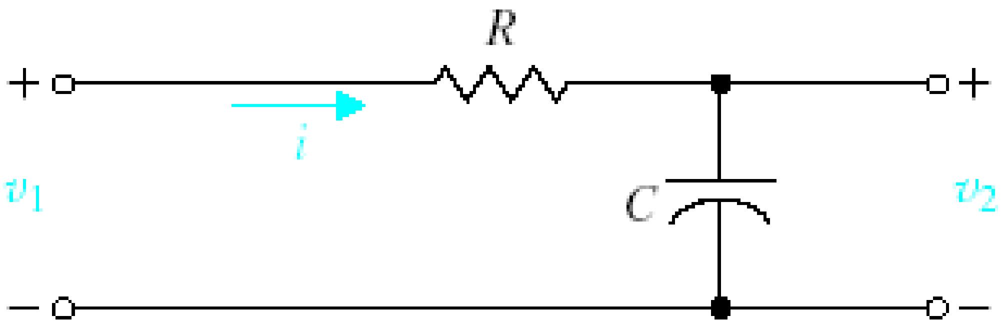
$$s2 = -\left(\zeta\cdot\omega_n\right) - j\cdot\omega_n\cdot\sqrt{1 - \zeta^2}$$

The Laplace Transform



Response of the spring-mass-damper system.

The Transfer Function of Linear Systems



An RC network.

$$V_1(s) = \left(R + \frac{1}{Cs}\right) \cdot I(s)$$

$$Z_1(s) = R$$

$$V_2(s) = \left(\frac{1}{Cs}\right) \cdot I(s)$$

$$Z_2(s) = \frac{1}{Cs}$$

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

The Transfer Function of Linear Systems

Example 2.2

$$\frac{d^2}{dt^2}y(t) + 4\cdot\frac{d}{dt}y(t) + 3\cdot y(t) = 2\cdot r(t)$$

Initial Conditions: $Y(0) = 1$ $\quad \frac{d}{dt}y(0) = 0 \quad r(t) = 1$

The Laplace transform yields:

$$\left(s^2\cdot Y(s) - s\cdot y(0)\right) + 4\cdot(s\cdot Y(s) - y(0)) + 3Y(s) = 2\cdot R(s)$$

Since R(s)=1/s and y(0)=1, we obtain:

$$Y(s) = \frac{(s+4)}{\left(s^2 + 4s + 3\right)} + \frac{2}{s\cdot\left(s^2 + 4s + 3\right)}$$

The partial fraction expansion yields:

$$Y(s) = \left[\frac{\frac{3}{2}}{(s+1)} + \frac{\frac{-1}{2}}{(s+3)}\right] + \left[\frac{-1}{(s+1)} + \frac{\frac{1}{3}}{(s+3)}\right] + \frac{\frac{2}{3}}{s}$$

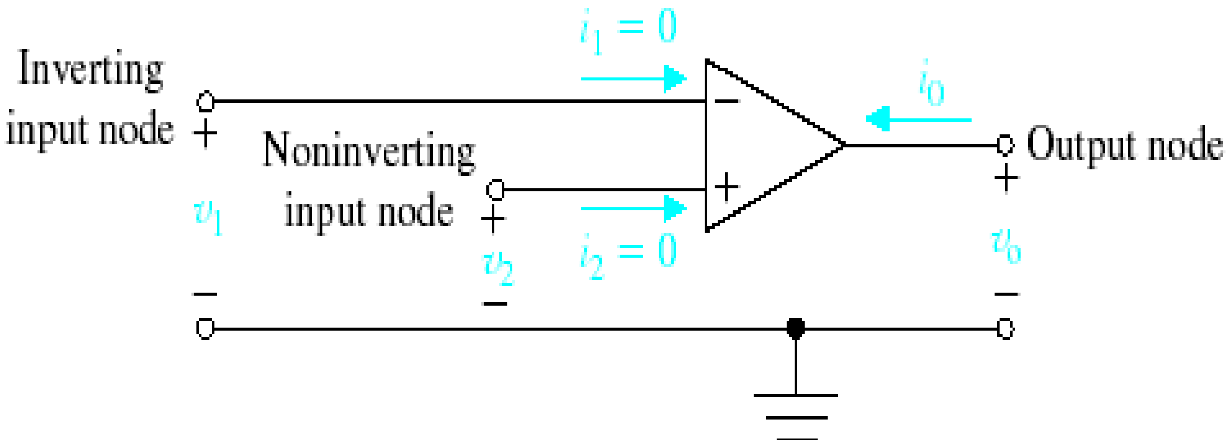
Therefore the transient response is:

$$y(t) = \left(\frac{3}{2}\cdot e^{-t} - \frac{1}{2}\cdot e^{-3\cdot t}\right) + \left(-1e^{-t} + \frac{1}{3}\cdot e^{-3t}\right) + \frac{2}{3}$$

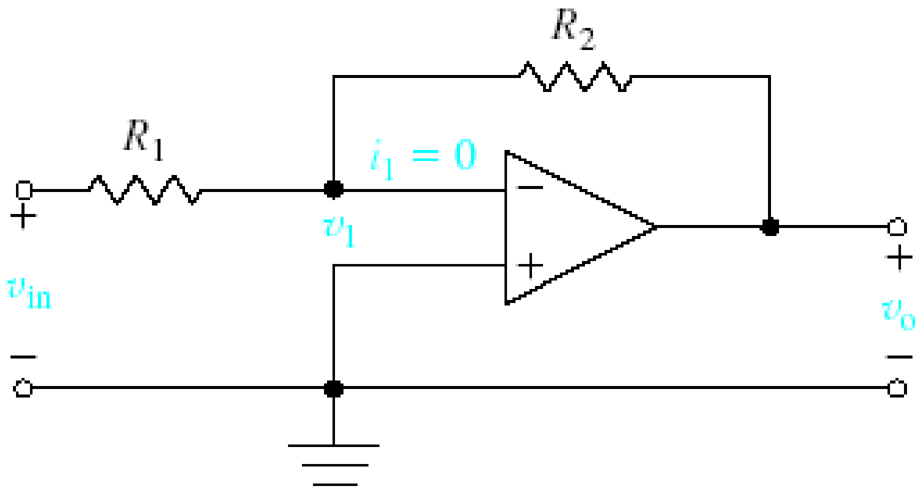
The steady-state response is:

$$\lim_{t\to\infty} y(t) = \frac{2}{3}$$

The Transfer Function of Linear Systems

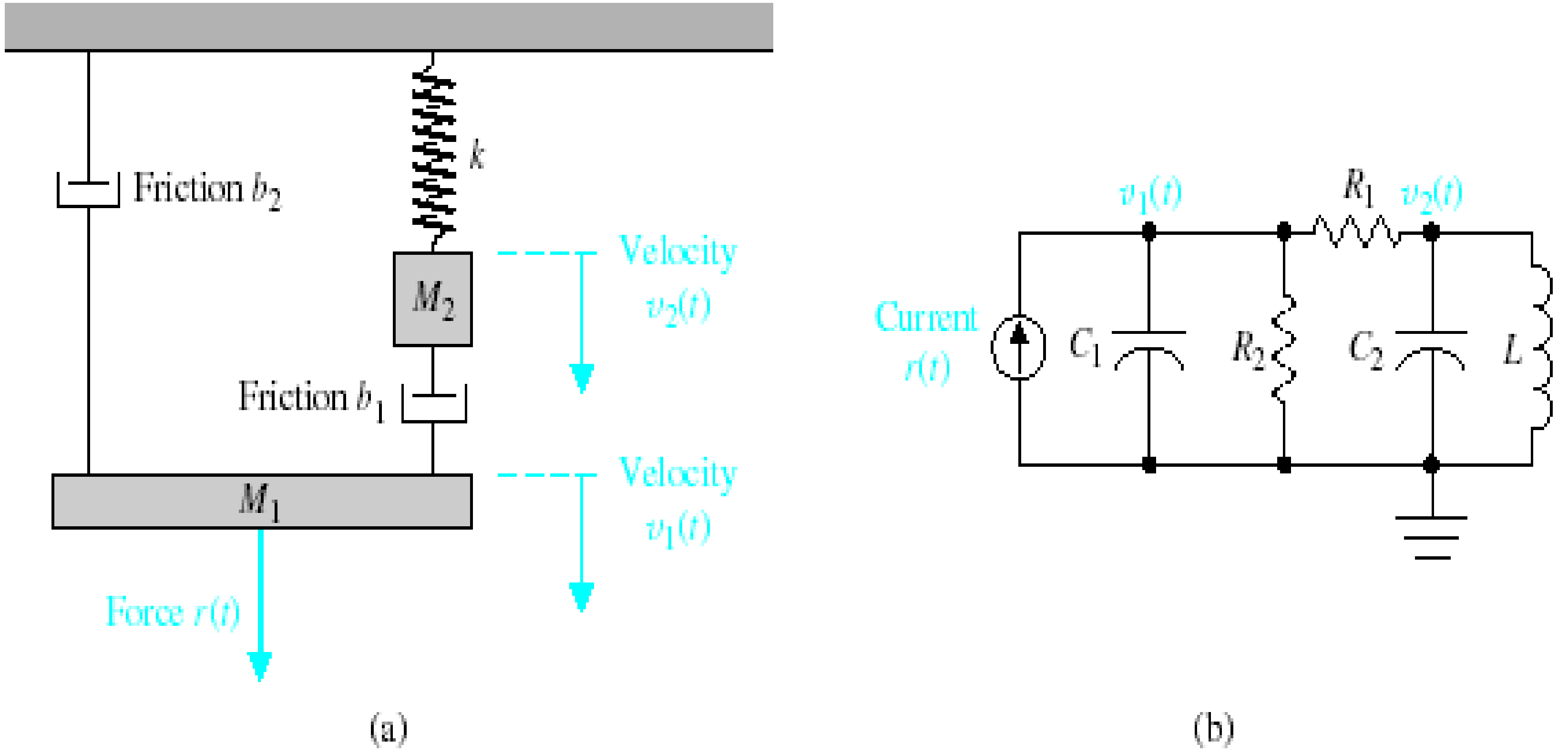


The ideal op-amp



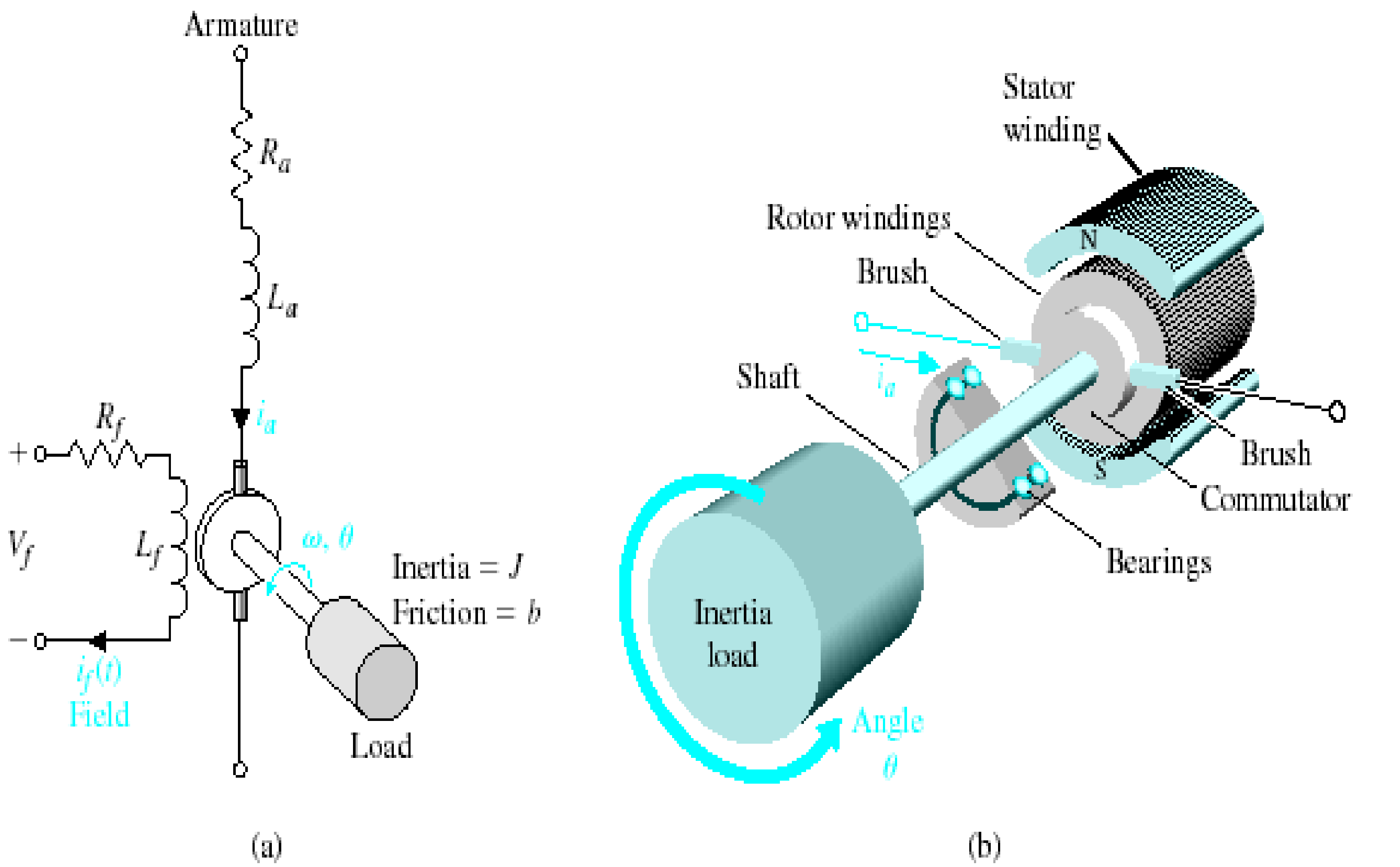
An inverting amplifier operating with ideal conditions.

The Transfer Function of Linear Systems



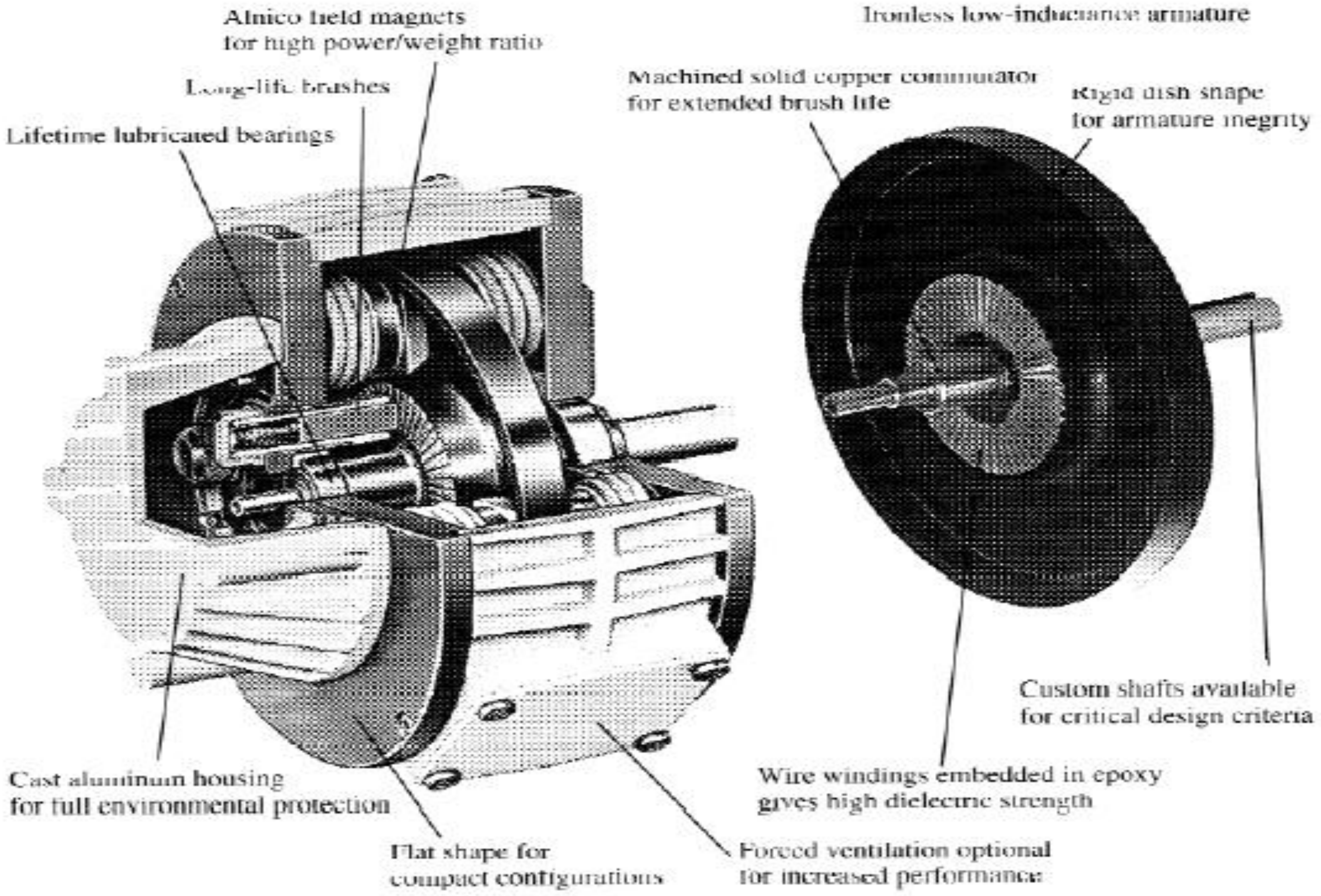
(a) Two-mass mechanical system. (b) Two-node electric circuit analog C1 = M1, C2 = M2, L = 1/k, R1 = 1/b1, R2 = 1/b2.

The Transfer Function of Linear Systems



A dc motor (a) wiring diagram and (b) sketch.

The Transfer Function of Linear Systems



A pancake dc motor with a flat-wound armature and a permanent magnet rotor. These motors are capable of providing high torque with a low rotor inertia. A typical mechanical time constant is in the range of 15 ms. (Courtesy of Mavilor Motors.)

The Transfer Function of Linear Systems

$$\phi = K_f \cdot i_f$$

$$T_m = K_1 \cdot K_f \cdot i_f(t) \cdot i_a(t)$$

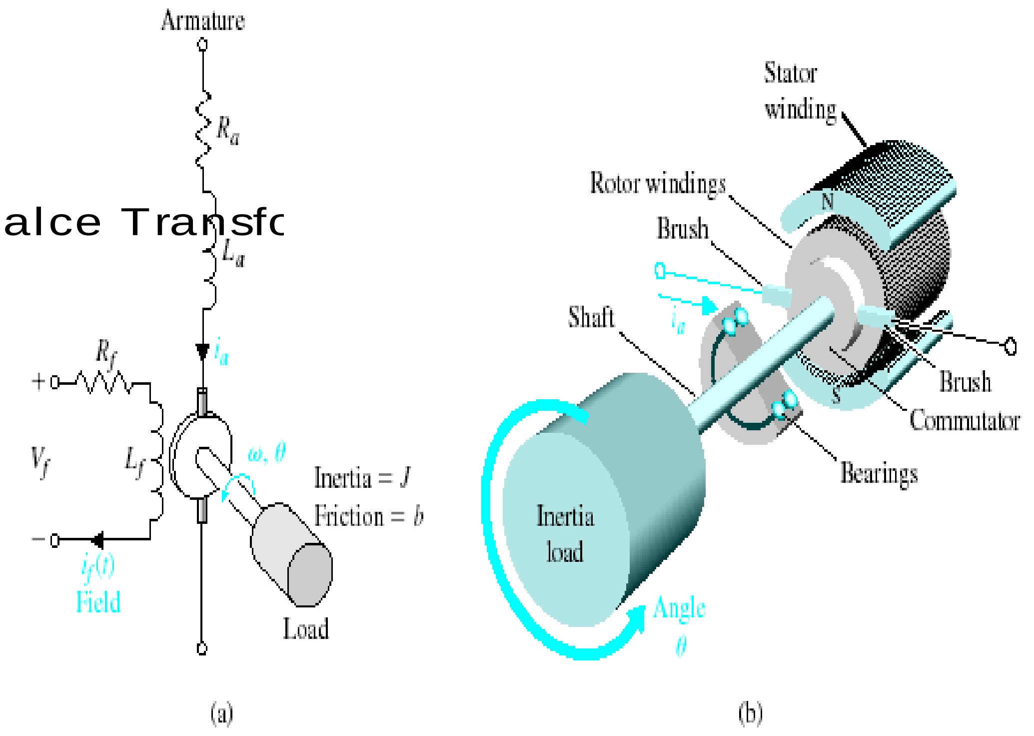
field controled motor - Lapalce Transfc

$$T_m(s) = \left(K_1 \cdot K_f \cdot I_a\right) \cdot I_f(s)$$

$$V_f(s) = \left(R_f + L_f \cdot s\right) \cdot I_f(s)$$

$$T_m(s) = T_L(s) + T_d(s)$$

$$T_L(s) = J \cdot s^2 \cdot \theta(s) + b \cdot s \cdot \theta(s)$$



rearranging equations

$$T_L(s) = T_m(s) - T_d(s)$$

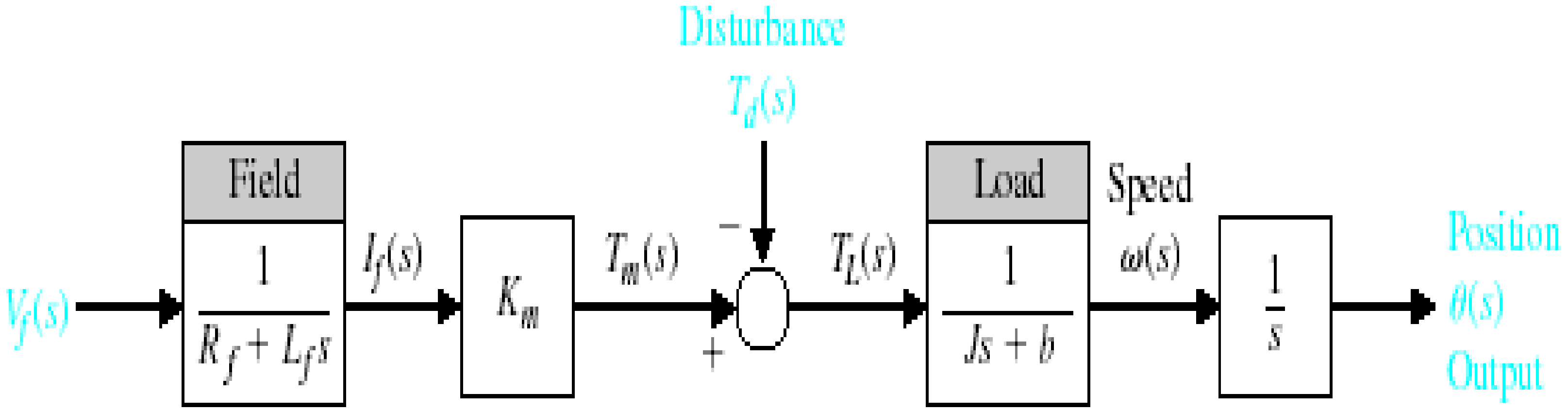
$$T_m(s) = K_m \cdot I_f(s)$$

$$I_f(s) = \frac{Vf(s)}{R_f + L_f \cdot s}$$

$$Td(s) = 0$$

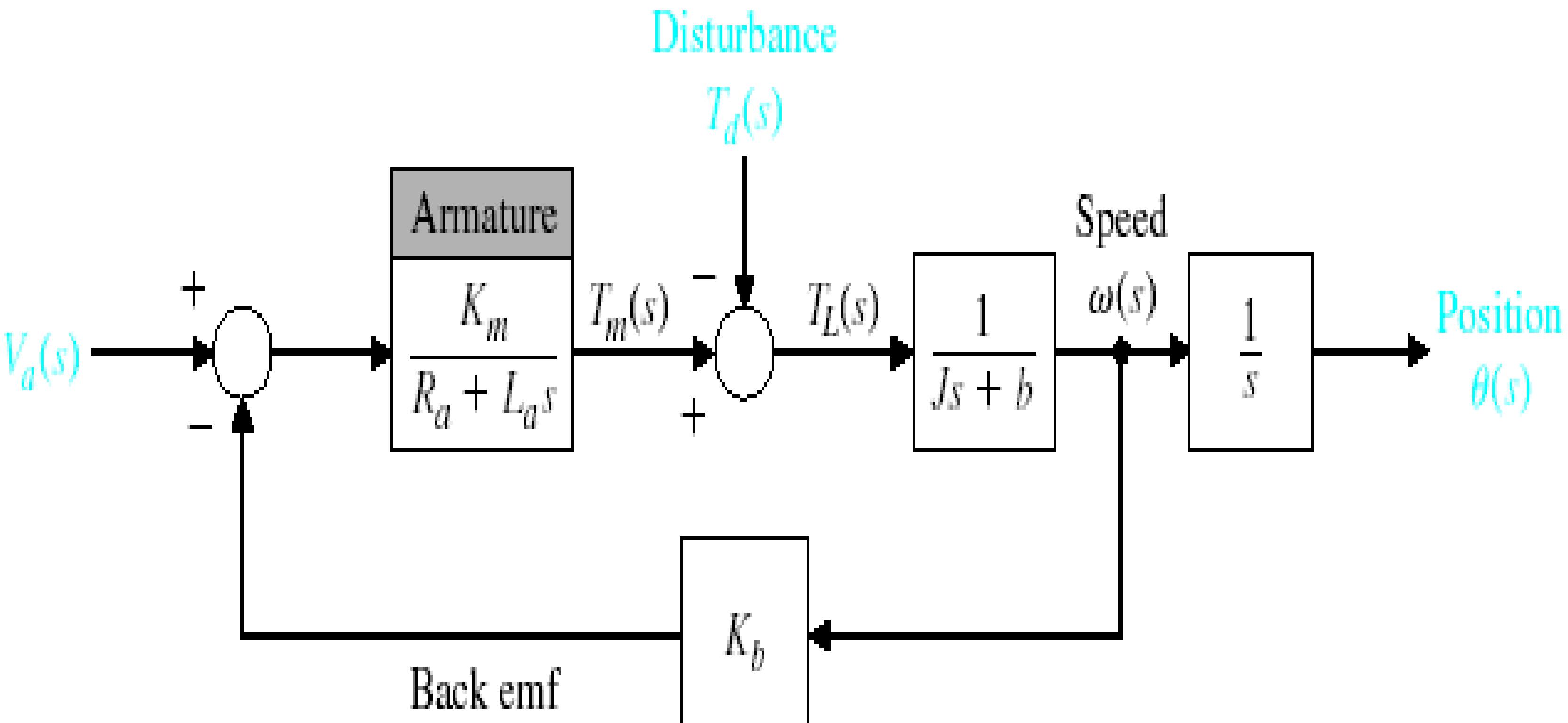
$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s \cdot (J \cdot s + b) \cdot \left(L_f \cdot s + R_f\right)}$$

The Transfer Function of Linear Systems



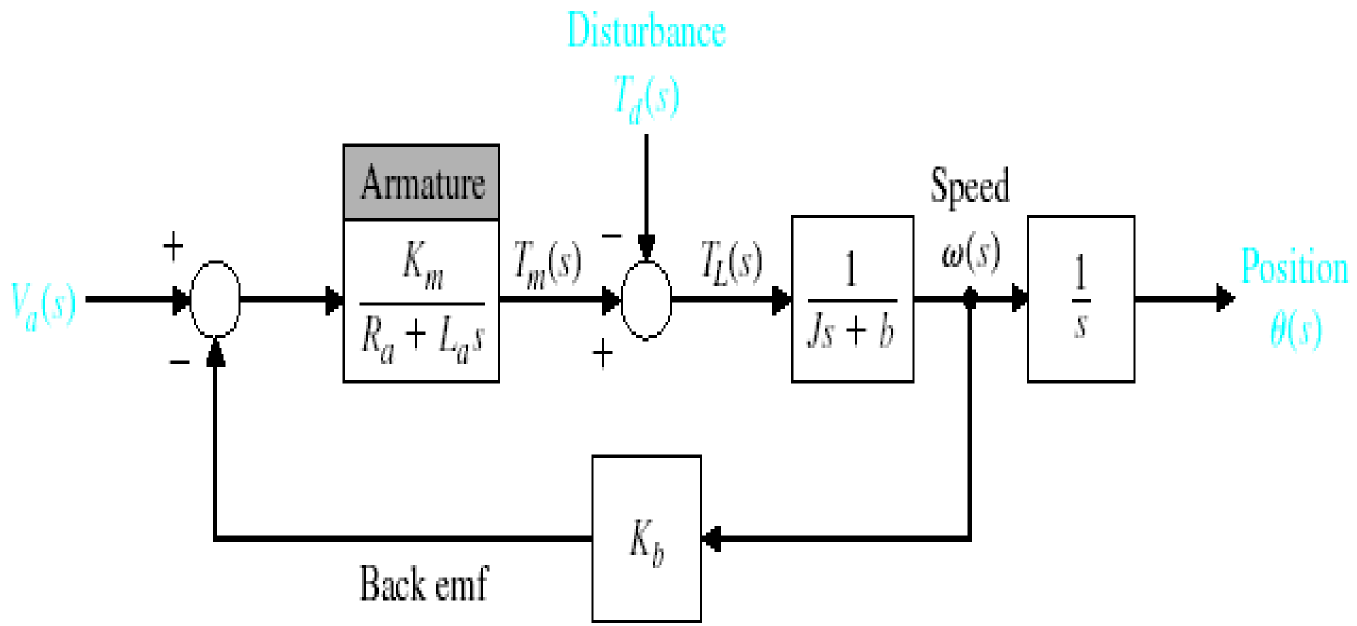
Block diagram model of field-controlled dc motor.

The Transfer Function of Linear Systems

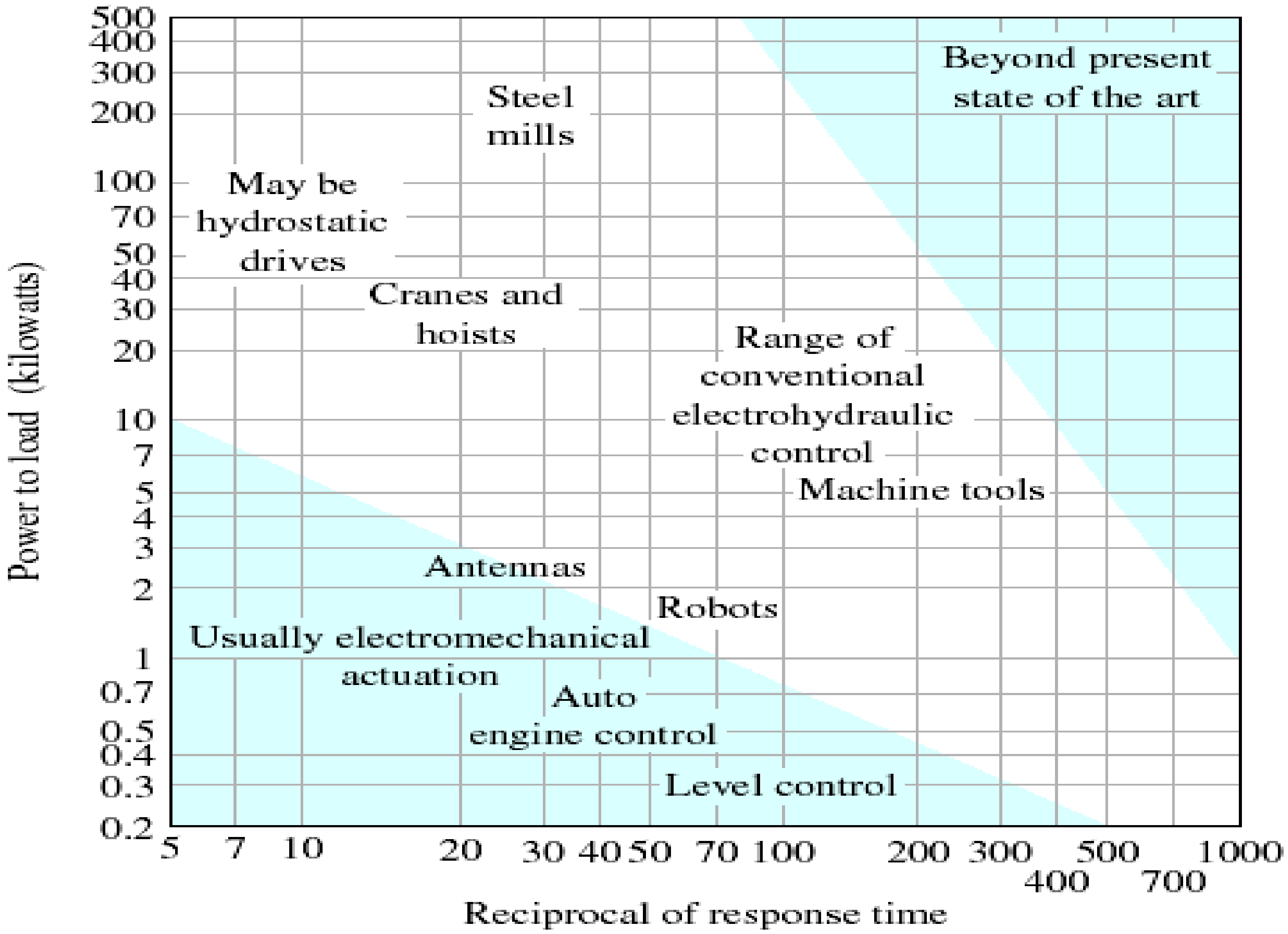


Armature-controlled dc motor.

The Transfer Function of Linear Systems

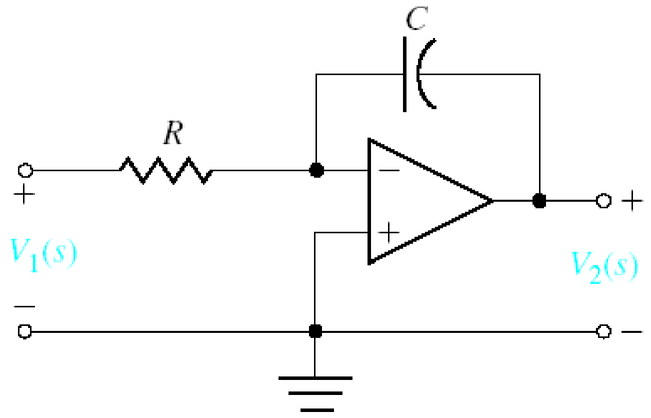


The Transfer Function of Linear Systems

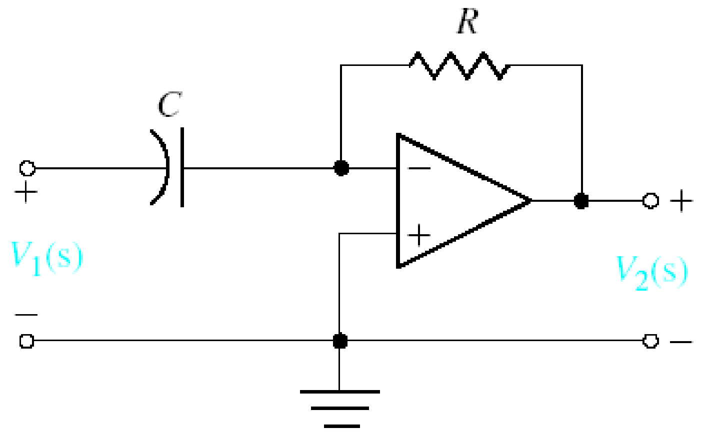


Range of control response time and power to load for electromechanical and electrohydraulic devices.

The Transfer Function of Linear Systems

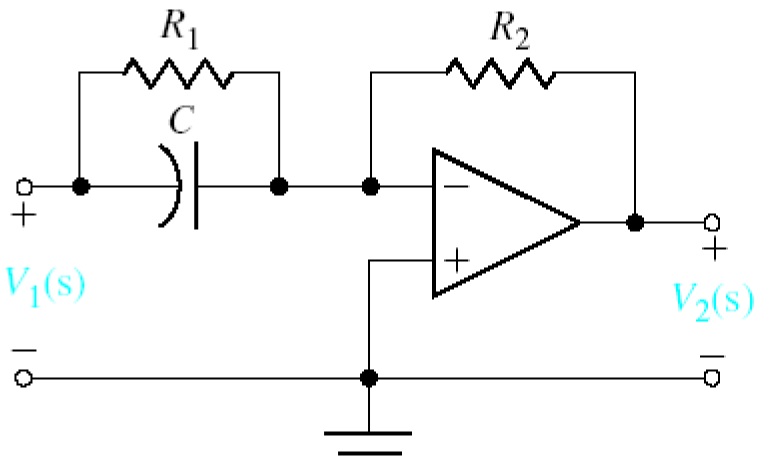


$$\frac{V_2(s)}{V_1(s)} = \frac{-1}{RCs}$$

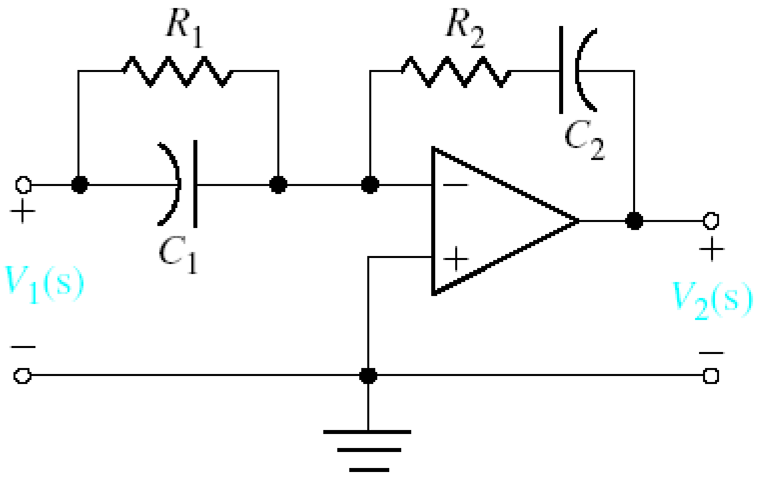


$$\frac{V_2(s)}{V_1(s)} = -RCs$$

The Transfer Function of Linear Systems

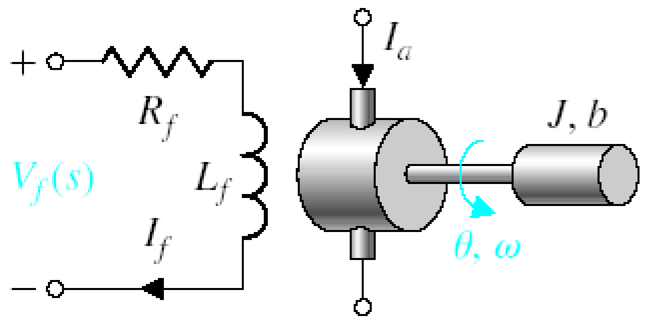


$$\frac{V_2(s)}{V_1(s)} = \frac{R_2\left(R_1 \cdot C \cdot s + 1\right)}{R_1}$$

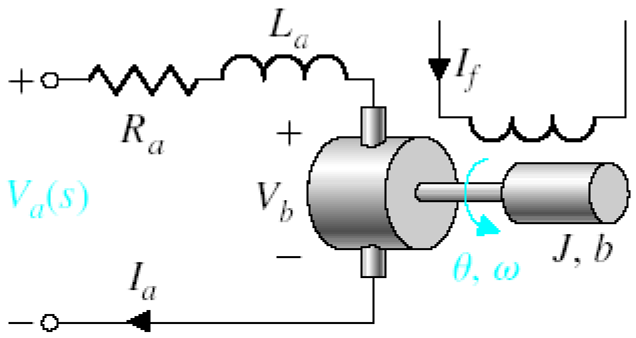


$$\frac{V_2(s)}{V_1(s)} = \frac{-\left(R_1 \cdot C_1 \cdot s + 1\right)\left(R_2 \cdot C_2 \cdot s + 1\right)}{R_1 \cdot C_2 \cdot s}$$

The Transfer Function of Linear Systems

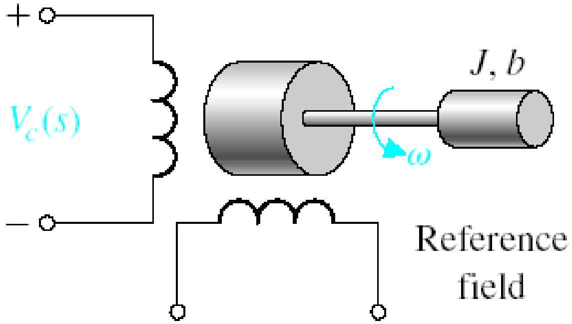


$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s\cdot(J\cdot s + b)\left(L_f\cdot s + R_f\right)}$$



$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s\cdot\left[\left(R_a + L_a\cdot s\right)(J\cdot s + b) + K_b\cdot K_m\right]}$$

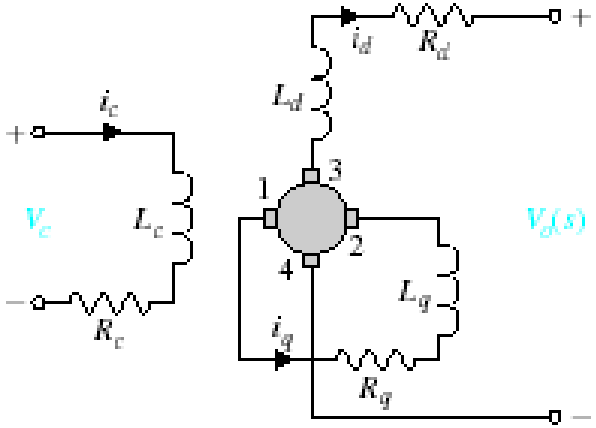
The Transfer Function of Linear Systems



$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau \cdot s + 1)}$$

$$\tau = \frac{J}{(b - m)}$$

m = slope of linearized torque-speed curve (normally negative)



$$\frac{V_o(s)}{V_c(s)} = \frac{\left(\frac{K}{R_c \cdot R_q}\right)}{(s \cdot \tau_c + 1) \cdot (s \cdot \tau_q + 1)}$$

$$\tau_c = \frac{L_c}{R_c} \qquad \tau_q = \frac{L_q}{R_q}$$

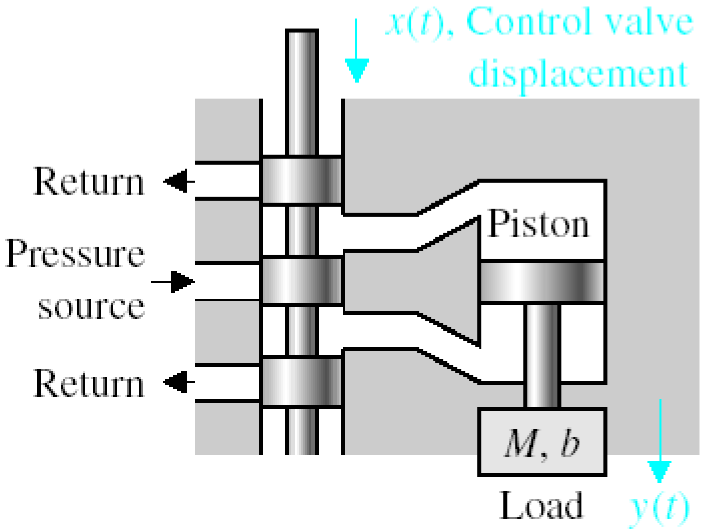
For the unloaded case:

$i_d = 0 \qquad \tau_c = \tau_q$

$0.05s < \tau_c < 0.5s$

$V_{12} = V_q \qquad V_{34} = V_d$

The Transfer Function of Linear Systems



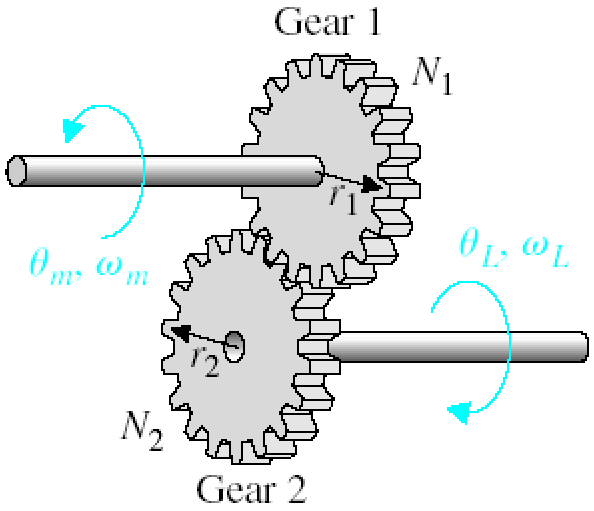
$$\frac{Y(s)}{X(s)} = \frac{K}{s(Ms + B)}$$

$$K = \frac{A \cdot k_x}{k_p} \qquad B = \left(b + \frac{A^2}{k_p}\right)$$

$$k_x = \frac{d}{dx} g \qquad k_p = \frac{d}{dP} g$$

$$g = g(x, P) = \text{flow}$$

A = area of piston



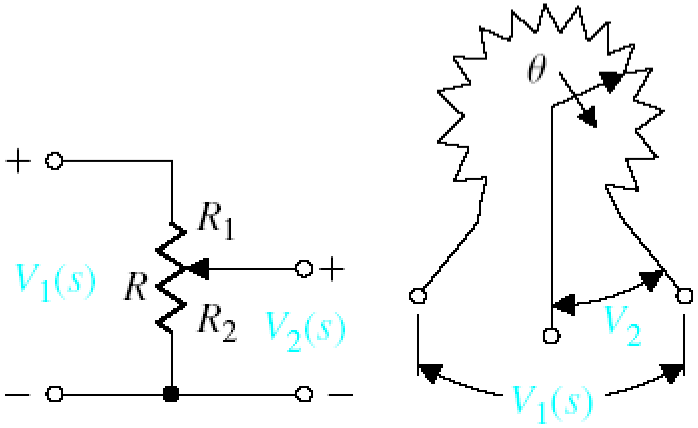
Gear Ratio = n = N1/N2

$$N_2 \cdot \theta_L = N_1 \cdot \theta_m$$

$$\theta_L = n \cdot \theta_m$$

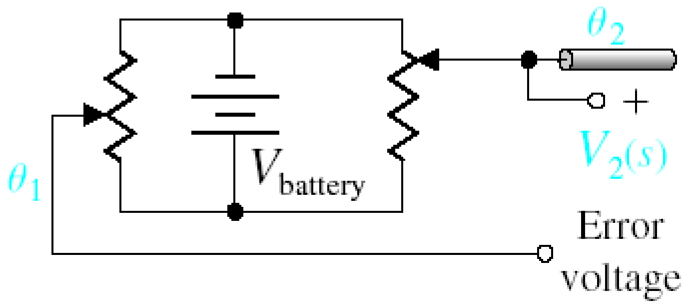
$$\omega_L = n \cdot \omega_m$$

The Transfer Function of Linear Systems



$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2}$$

$$\frac{R_2}{R} = \frac{\theta}{\theta_{max}}$$

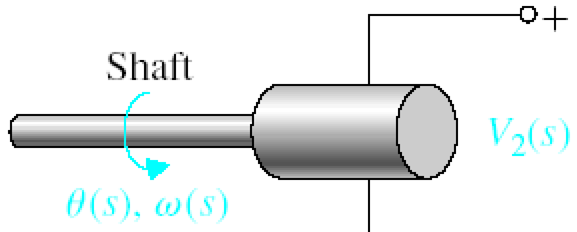


$$V_2(s) = k_s\left(\theta_1(s) - \theta_2(s)\right)$$

$$V_2(s) = k_s \cdot \theta_{error}(s)$$

$$k_s = \frac{V_{battery}}{\theta_{max}}$$

The Transfer Function of Linear Systems



$$V_2(s) = K_t \cdot \omega(s) = K_t \cdot s \cdot \theta(s)$$

K_t = constant

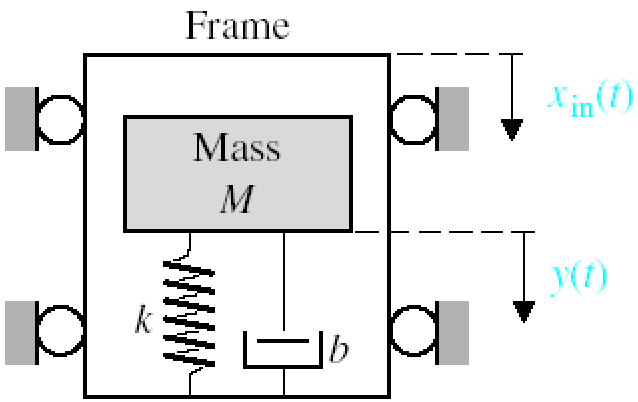
$$\frac{V_2(s)}{V_1(s)} = \frac{k_a}{s \cdot \tau + 1}$$

Ro = output resistance
Co = output capacitance

$\tau = R_o \cdot C_o$ $\tau < 1s$

and is often negligible
for controller amplifie

The Transfer Function of Linear Systems

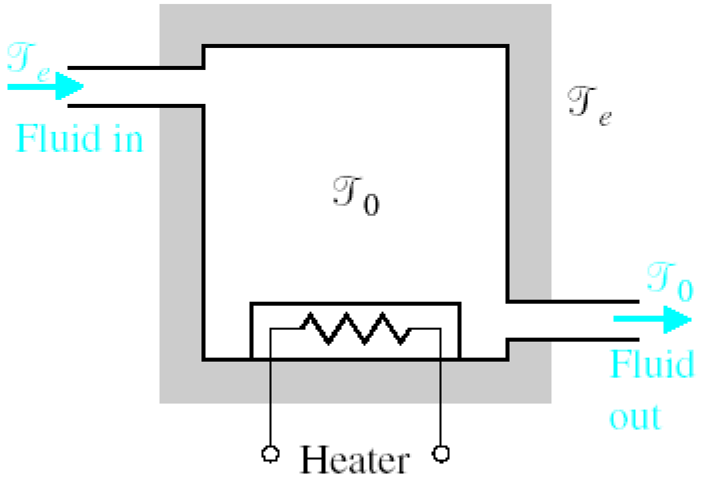


$x_o(t) = y(t) - x_{in}(t)$

$$\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + \left(\frac{b}{M}\right)\cdot s + \frac{k}{M}}$$

For low frequency oscillations, where $\omega < \omega_n$

$$\frac{X_o(j\cdot\omega)}{X_{in}(j\cdot\omega)} = \frac{\omega^2}{\frac{k}{M}}$$

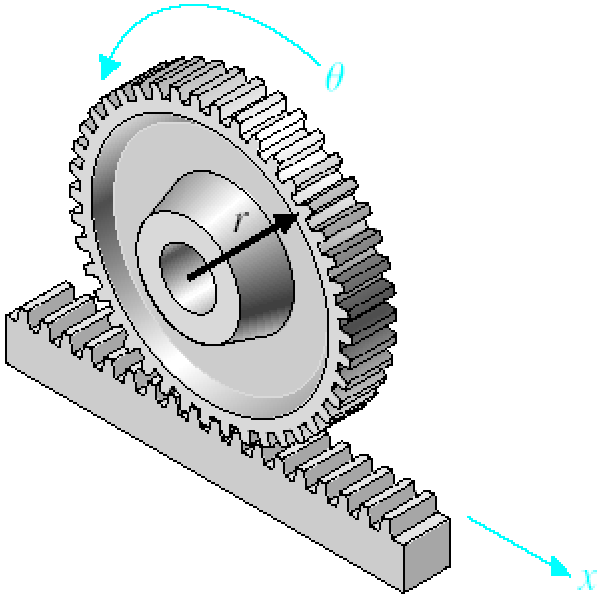


$$\frac{T(s)}{q(s)} = \frac{1}{C_t\cdot s + \left(Q\cdot S + \frac{1}{R}\right)}$$

$T = T_o - T_e$ = temperature difference due to thermal proc

C_t = thermal capacitance
Q = fluid flow rate = constant
S = specific heat of water
R_t = thermal resistance of insulation
$q(s)$ = rate of heat flow of heating element

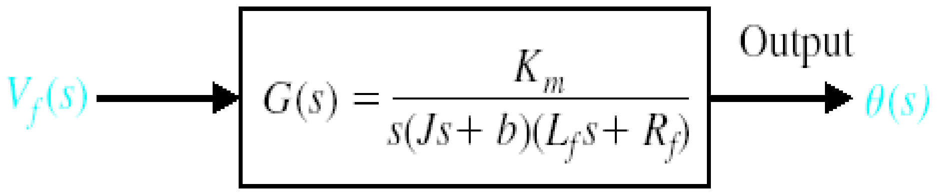
The Transfer Function of Linear Systems



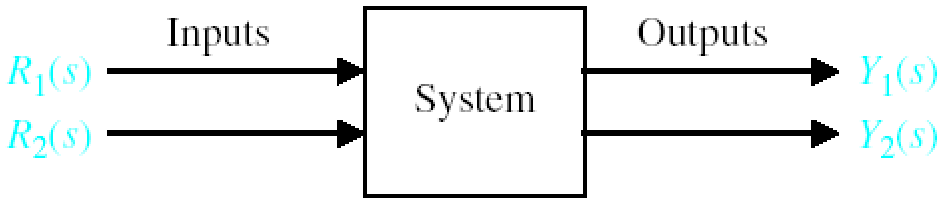
$x = r \cdot \theta$

converts radial motion to linear mo

Block Diagram Models

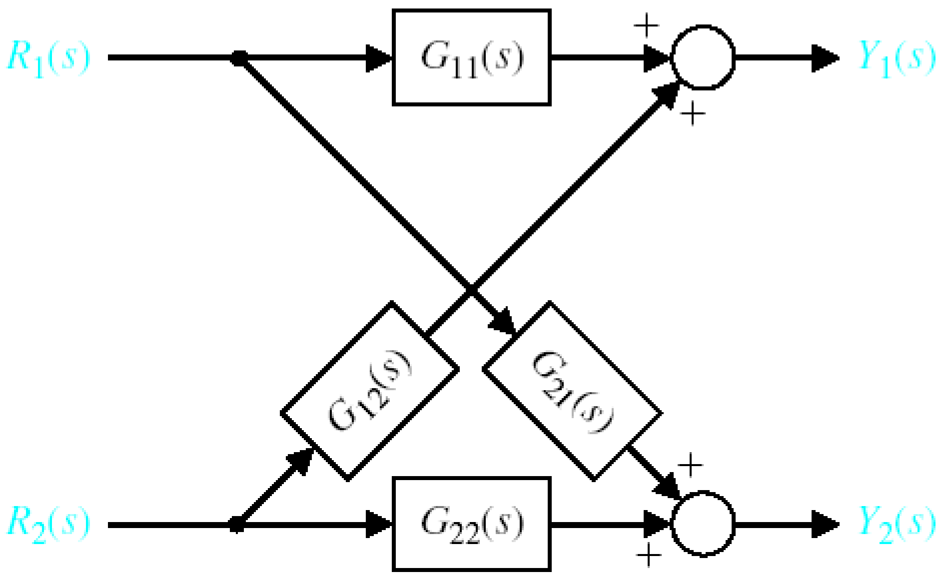


Block diagram of dc motor.



General block representation of two-input, two-output system.

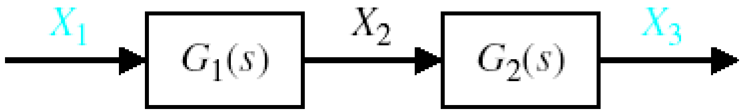
Block Diagram Models



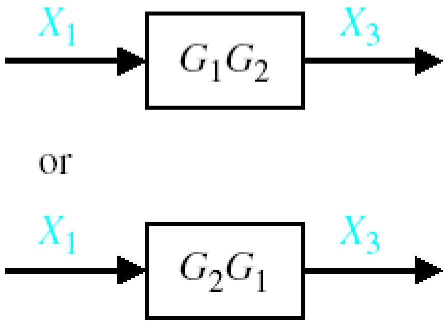
Block diagram of interconnected system.

Block Diagram Models

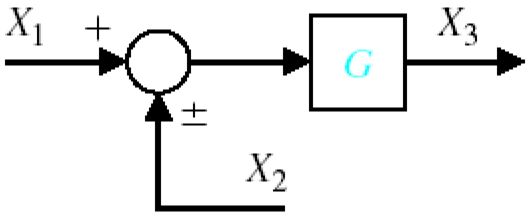
Original Diagram



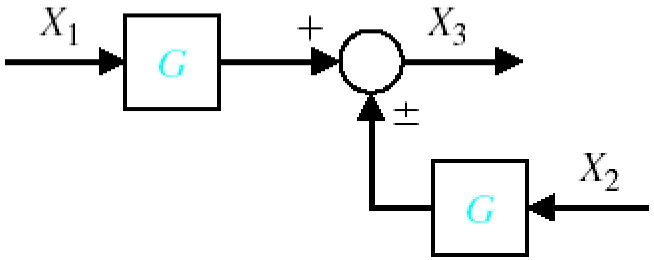
Equivalent Diagram



Original Diagram

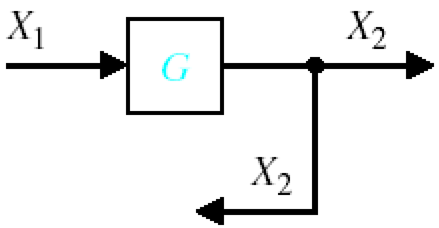


Equivalent Diagram

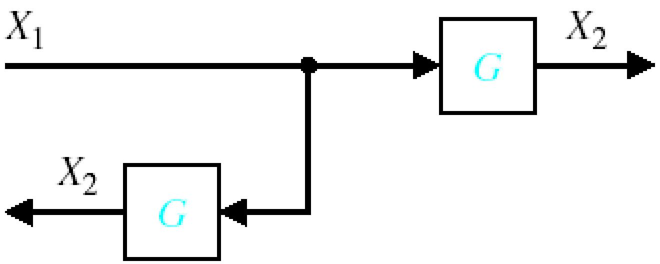


Block Diagram Models

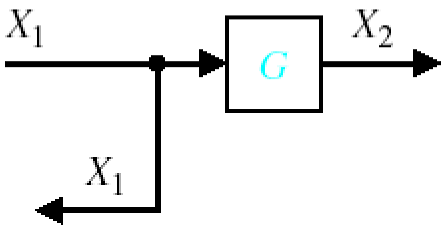
Original Diagram



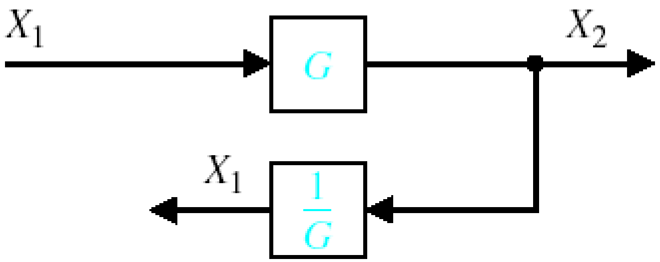
Equivalent Diagram



Original Diagram

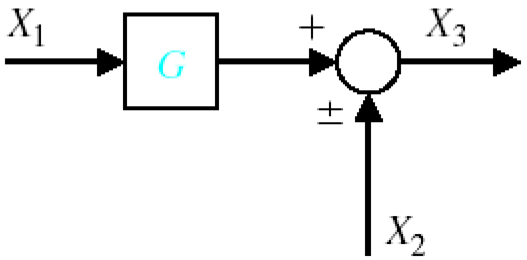


Equivalent Diagram

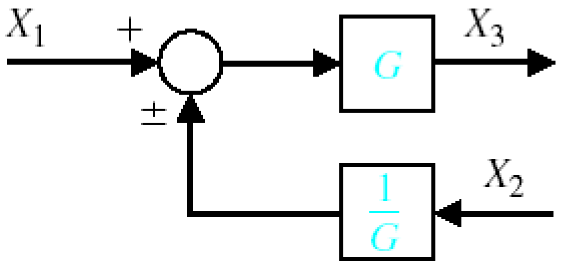


Block Diagram Models

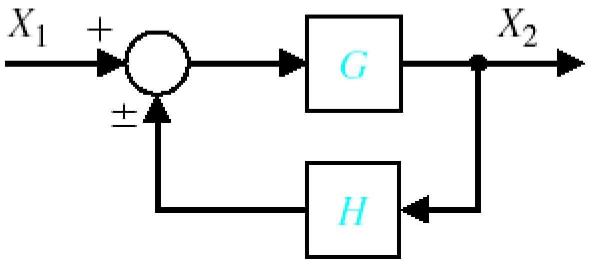
Original Diagram



Equivalent Diagram

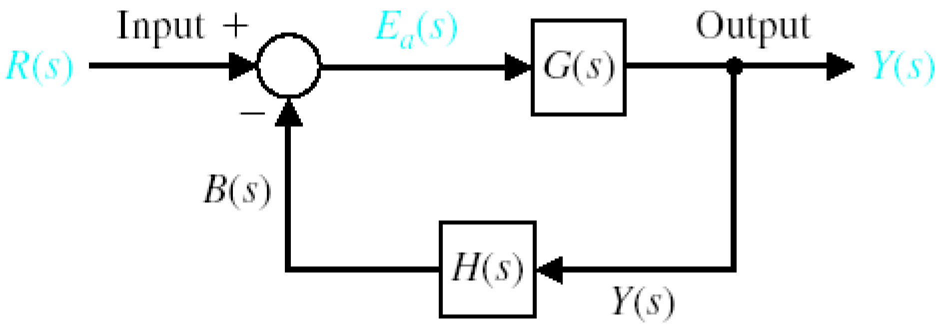


Original Diagram



Equivalent Diagram

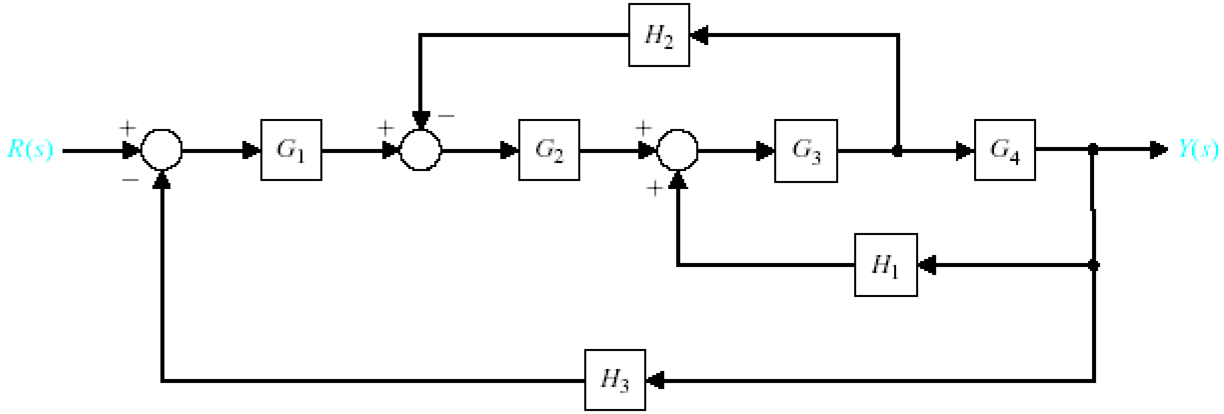
Block Diagram Models



Negative feedback control system.

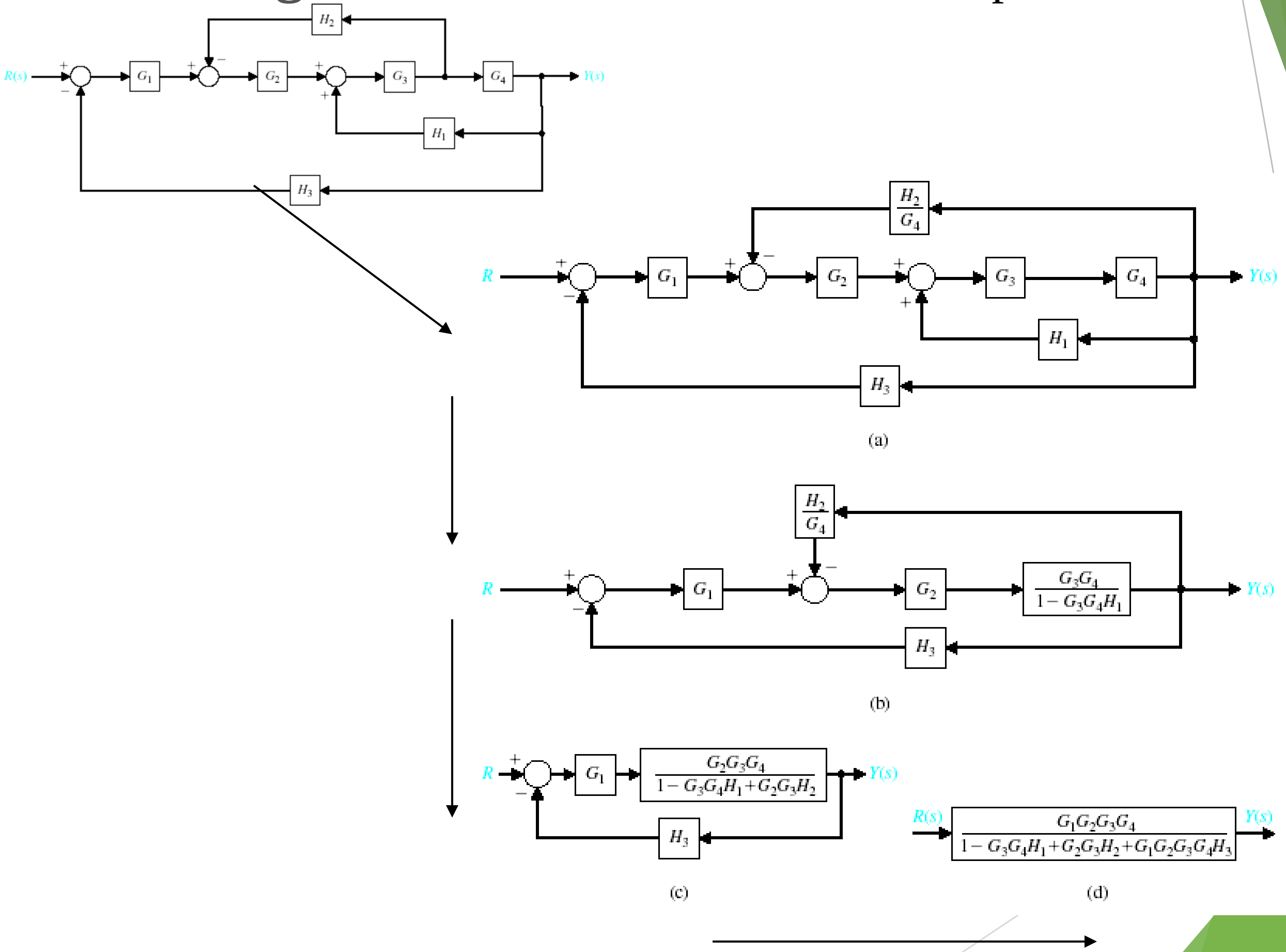
Block Diagram Models

Example 2.7



Block Diagram Models

Example 2.7

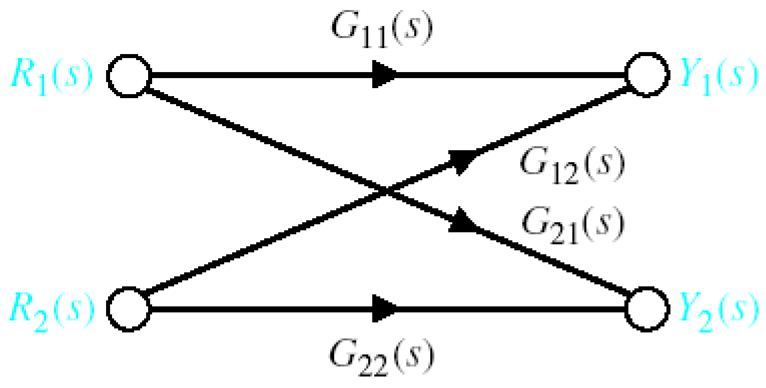


Signal-Flow Graph Models

Signal-flow graph of the dc motor.

For complex systems, the block diagram method can become difficult to complete. By using the signal-flow graph model, the reduction procedure (used in the block diagram method) is not necessary to determine the relationship between system variables.

Signal-Flow Graph Models

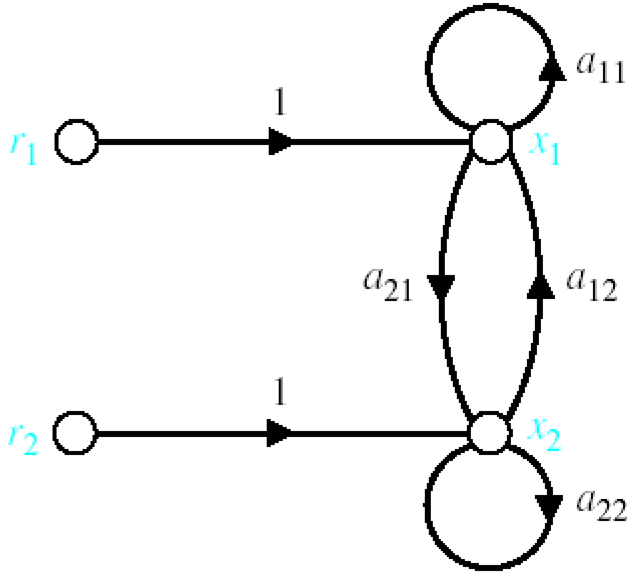


Signal-flow graph of interconnected system.

$$Y_1(s) = G_{11}(s) \cdot R_1(s) + G_{12}(s) \cdot R_2(s)$$

$$Y_2(s) = G_{21}(s) \cdot R_1(s) + G_{22}(s) \cdot R_2(s)$$

Signal-Flow Graph Models



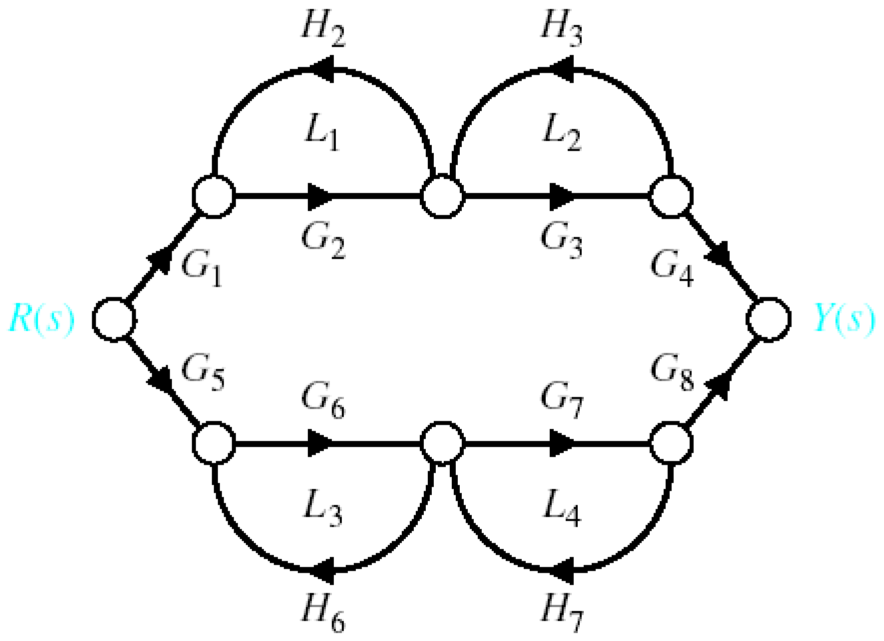
Signal-flow graph of two algebraic equations.

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + r_1 = x_1$$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + r_2 = x_2$$

Signal-Flow Graph Models

Example 2.8

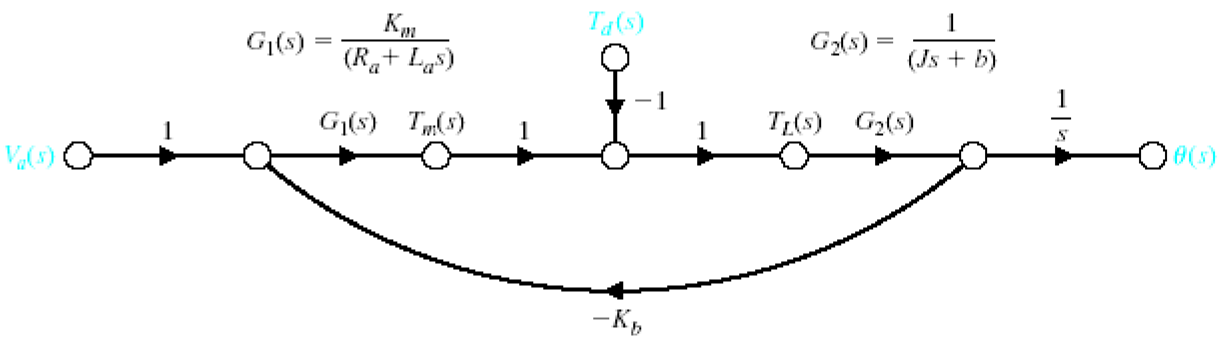


Two-path interacting system.

$$\frac{Y(s)}{R(s)} = \frac{\left[G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot \left(1 - L_3 - L_4\right)\right] + \left[G_5 \cdot G_6 \cdot G_7 \cdot G_8 \cdot \left(1 - L_1 - L_2\right)\right]}{1 - L_1 - L_2 - L_3 - L_4 + L_1 \cdot L_3 + L_1 \cdot L_4 + L_2 \cdot L_3 + L_2 \cdot L_4}$$

Signal-Flow Graph Models

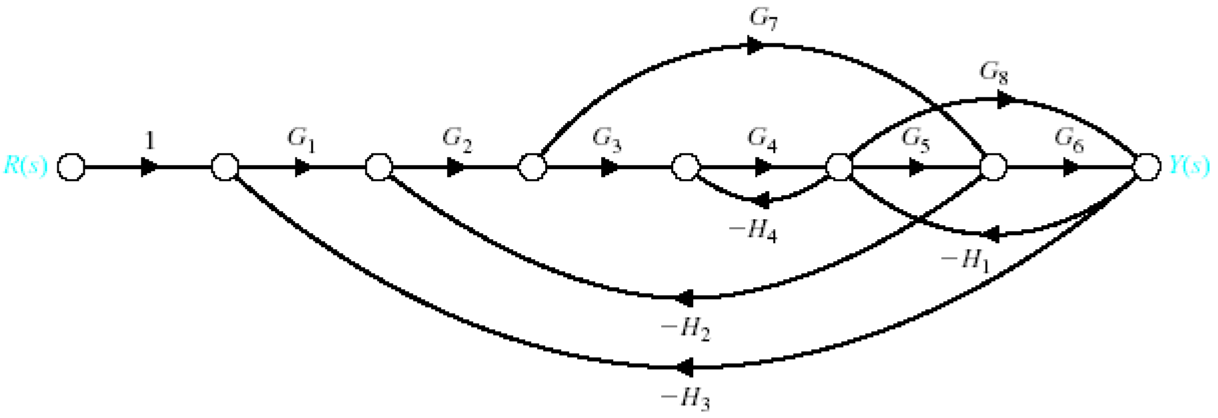
Example 2.10



The signal-flow graph of the armature-controlled dc motor.

$$\frac{Y(s)}{R(s)} = \frac{G_1 \cdot G_2 \cdot G_3 \cdot G_4}{1 + G_2 \cdot G_3 \cdot H_2 - G_3 \cdot G_4 \cdot H_1 + G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot H_3}$$

Signal-Flow Graph Models



Multiple-loop system.

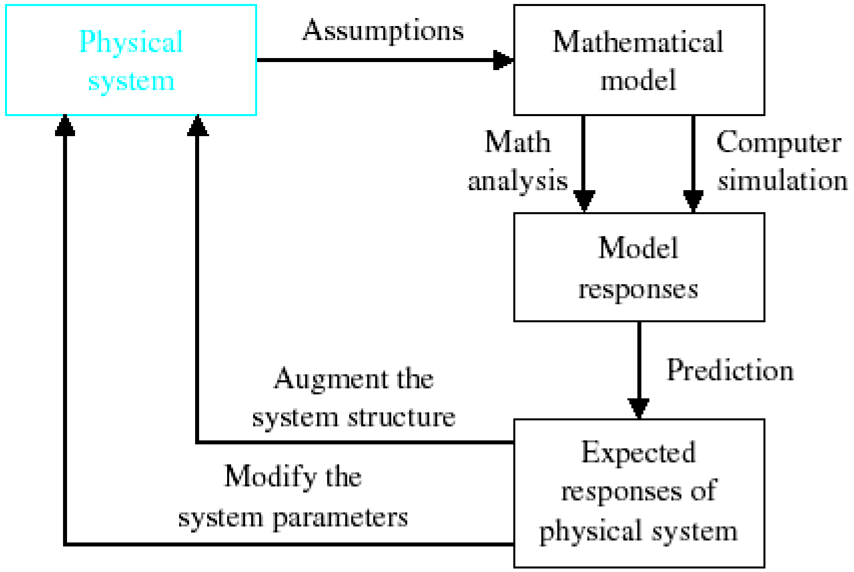
$$\frac{Y(s)}{R(s)} = \frac{P_1 + P_2 \cdot \Delta_2 + P_3}{\Delta}$$

$P_1 = G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_5 \cdot G_6$ $\quad$ $P_2 = G_1 \cdot G_2 \cdot G_7 \cdot G_6$ $\quad$ $P_3 = G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_8$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_5 \cdot L_7 + L_5 \cdot L_4 + L_3 \cdot L_4)$$

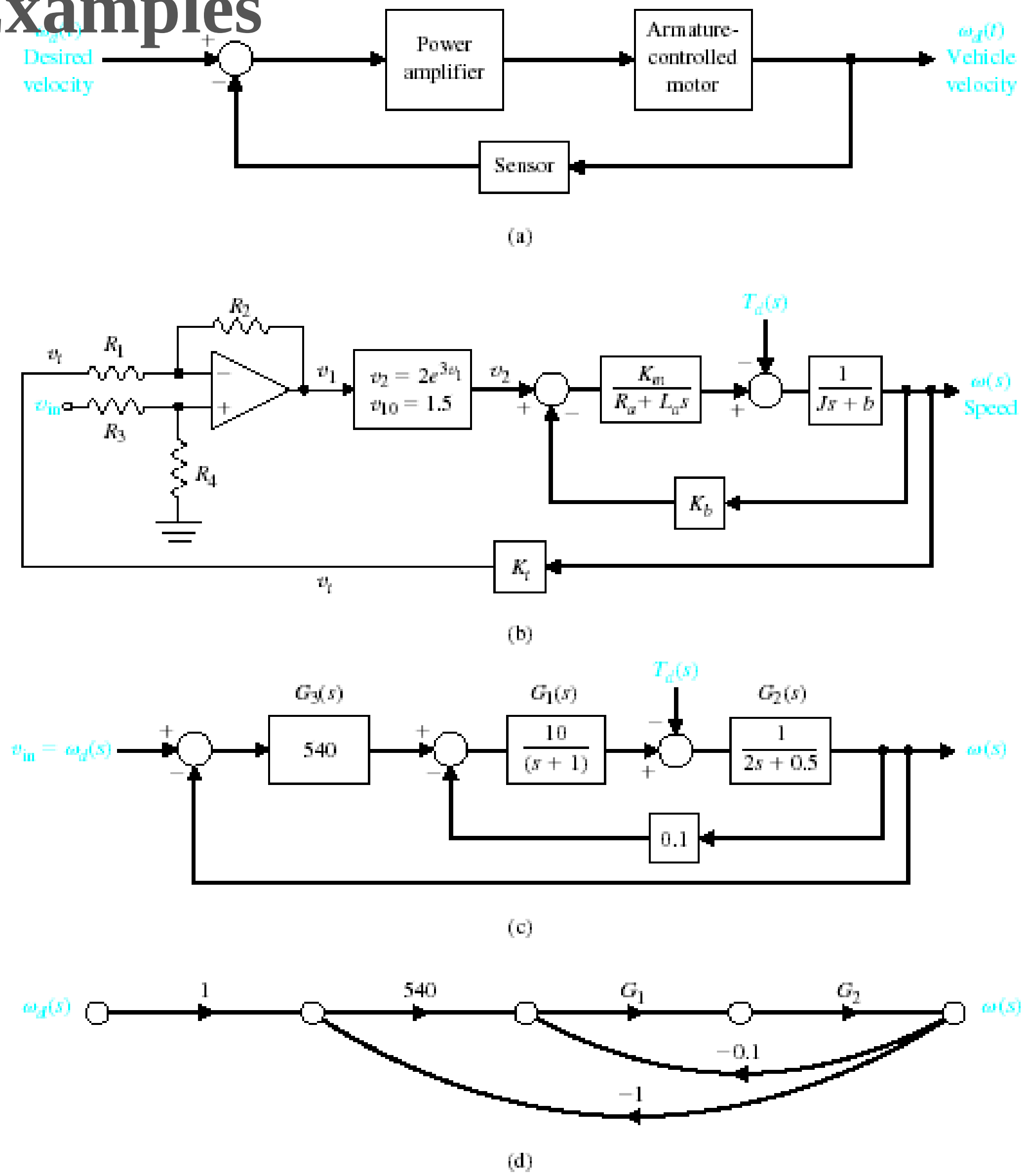
$\Delta_1 = \Delta_3 = 1$ $\quad$ $\Delta_2 = 1 - L_5 = 1 + G_4 \cdot H_4$

Design Examples



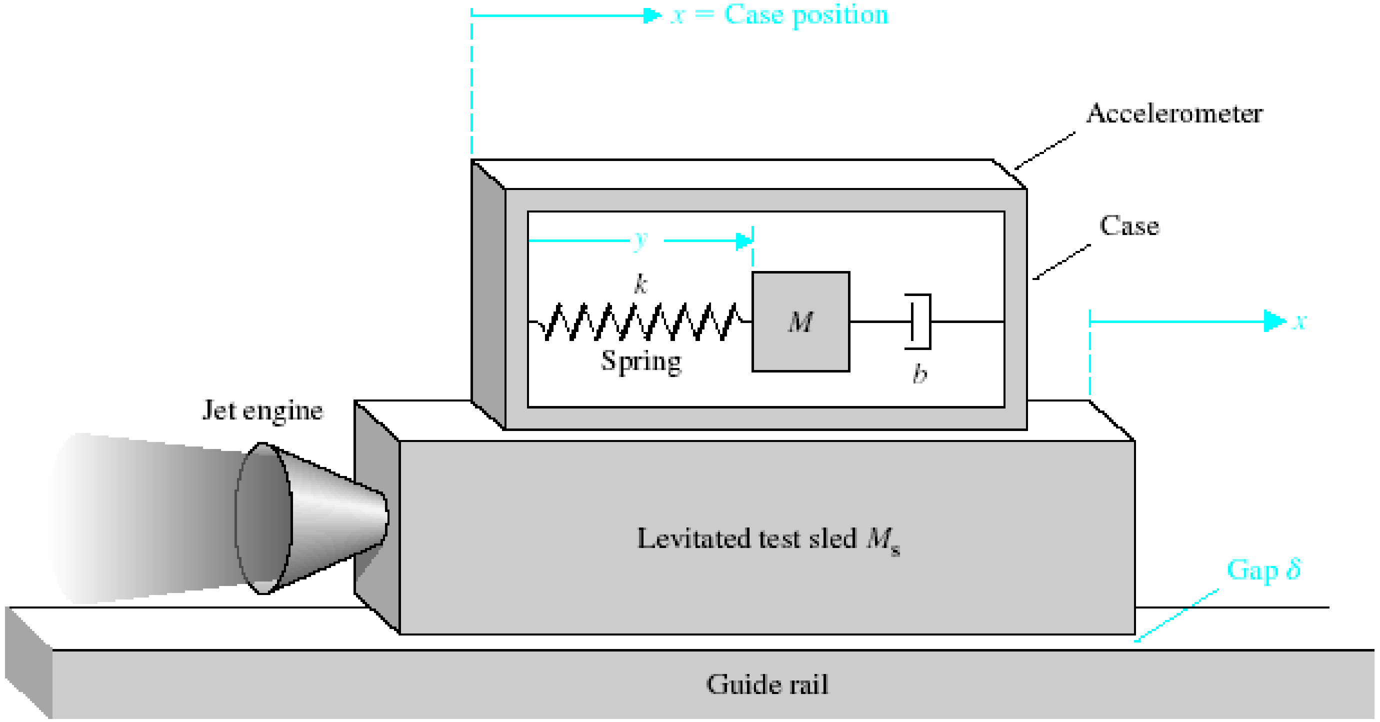
Analysis and design using a system model.

Design Examples



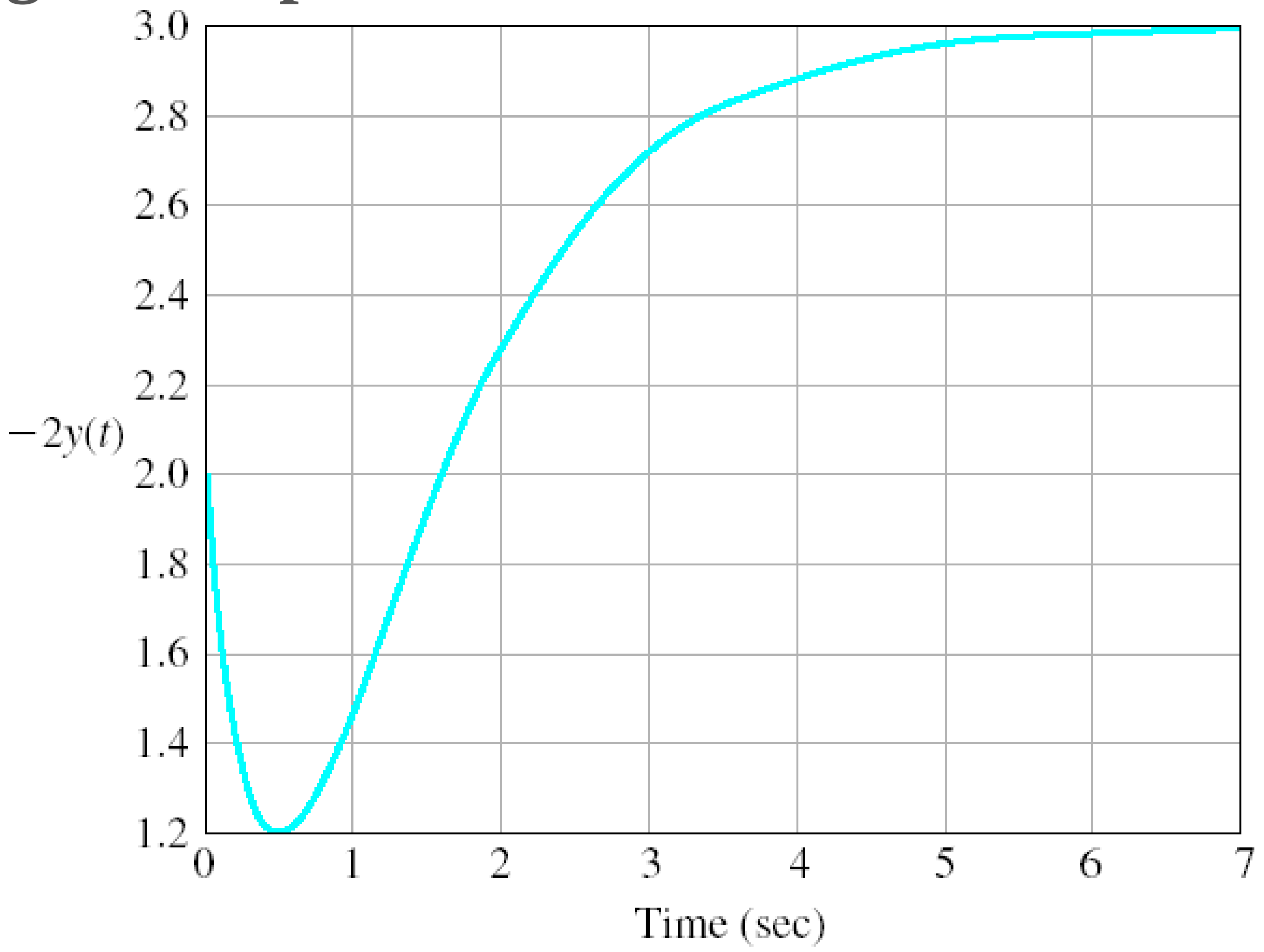
Speed control of an electric traction motor.

Design Examples



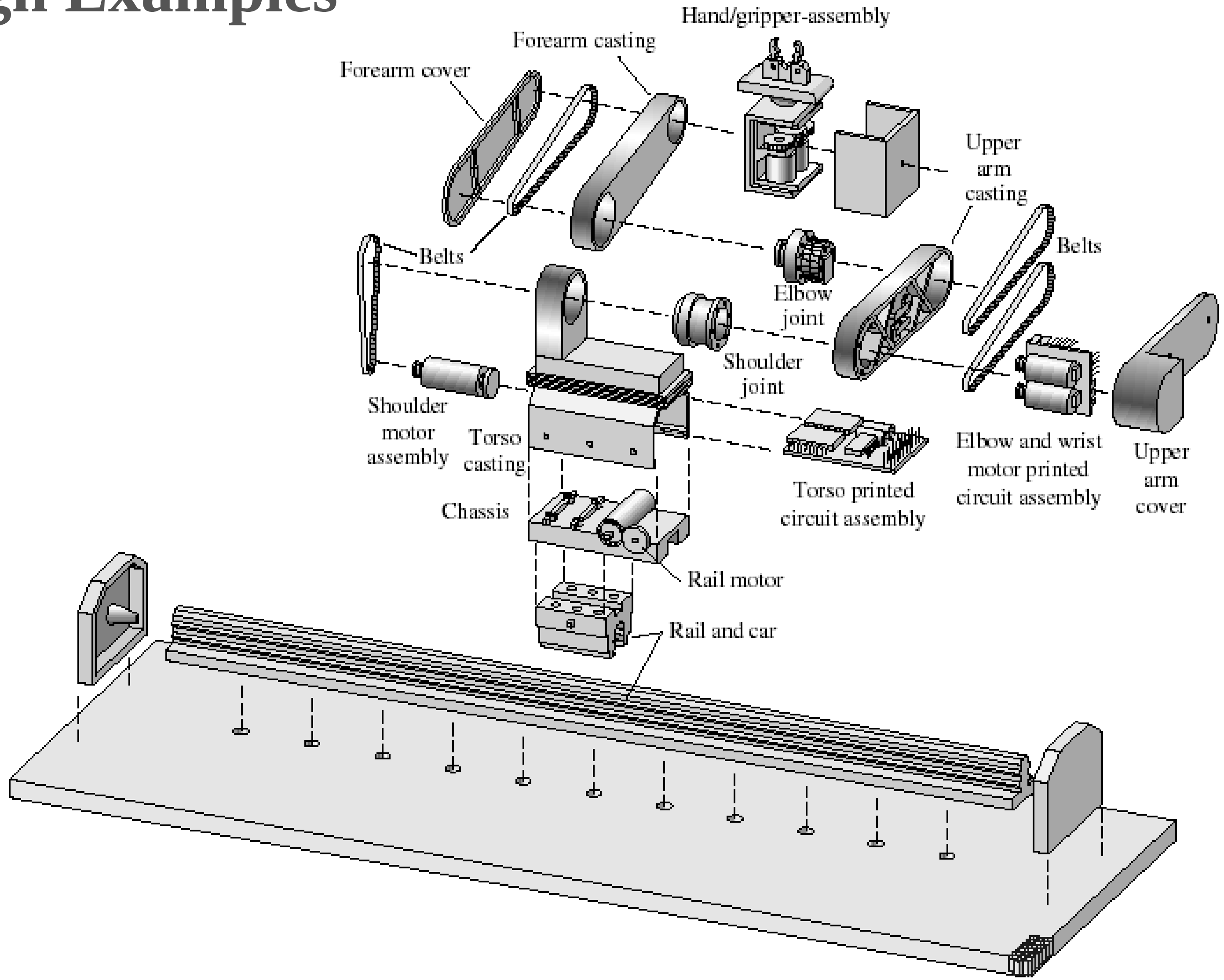
An accelerometer mounted on a jet-engine test sled.

Design Examples



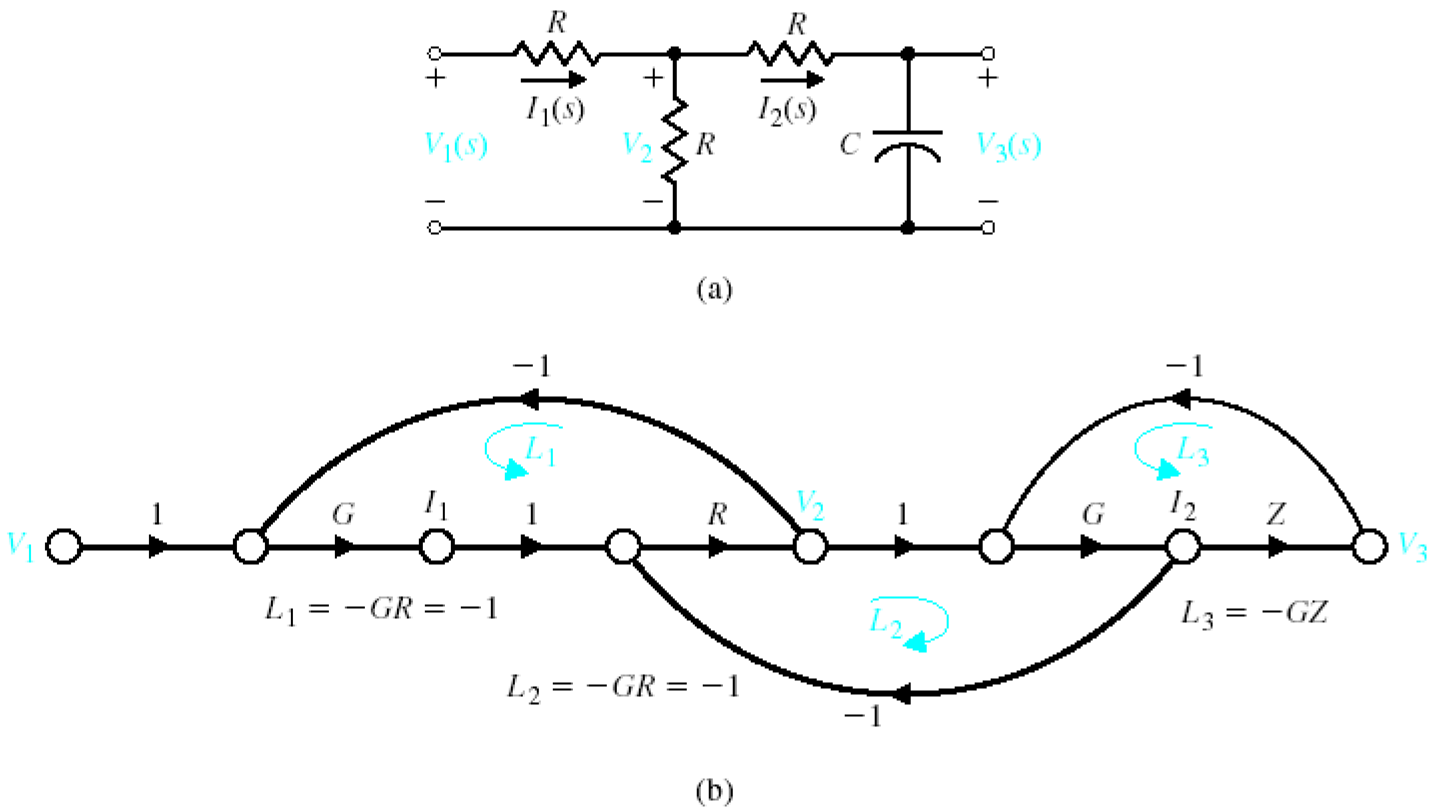
Accelerometer response.

Design Examples



Exploded view of the ORCA robot showing the components [15].
(Source: © Copyright 1993 Hewlett-Packard Company. Reproduced with permission.)

Design Examples



(a) Ladder network
and (b) its signal-flow graph.

The Simulation of Systems Using MATLAB

```
>>y0=0.15;
>>wn=sqrt(2);
>>zeta=1/(2*sqrt(2));
>>t=[0:0.1:10];
>>unforced
```

ω_n

ζ

unforced.m

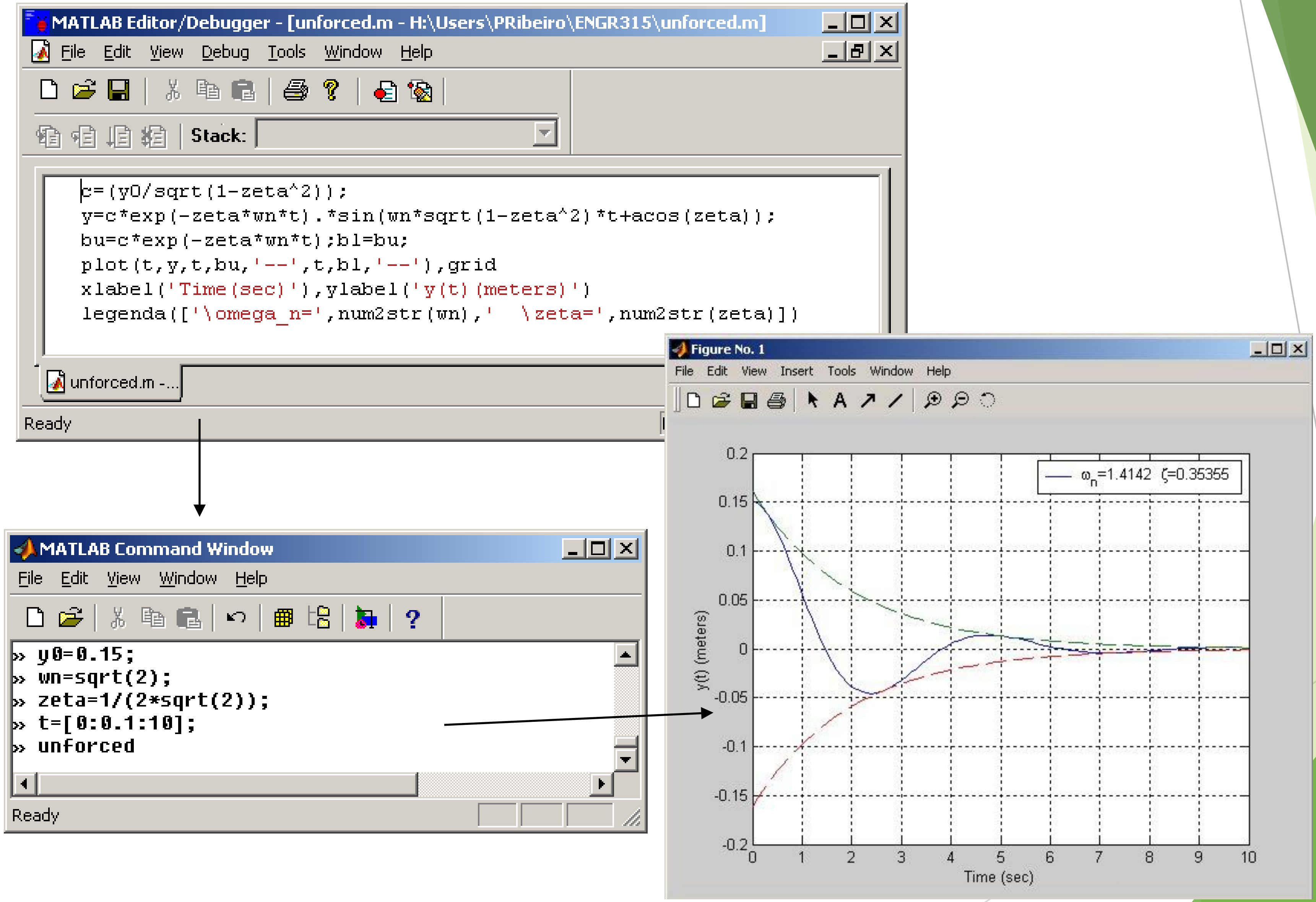
```
%Compute Unforced Response to an Initial Condition
%
c=(y0/sqrt(1-zeta^2));
y=c*exp(-zeta*wn*t).*sin(wn*sqrt(1-zeta^2)*t+acos(zeta));
%
bu=c*exp(-zeta*wn*t);bl=-bu;
%
plot(t,y,t,bu,'--',t,bl,'--'), grid
xlabel('Time (sec)'), ylabel('y(t) (meters)')
legend(['\omega_n=',num2str(wn),'     \zeta=',num2str(zeta)])
```

$y(0)/\sqrt{1-\zeta^2}$

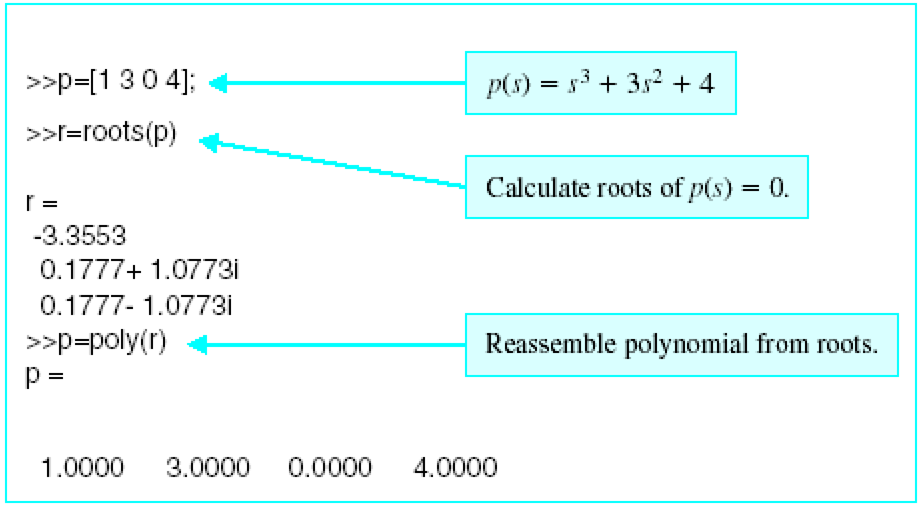
$e^{-\zeta\omega_n t}$ envelope

Script to analyze the spring-mass-damper.

The Simulation of Systems Using MATLAB

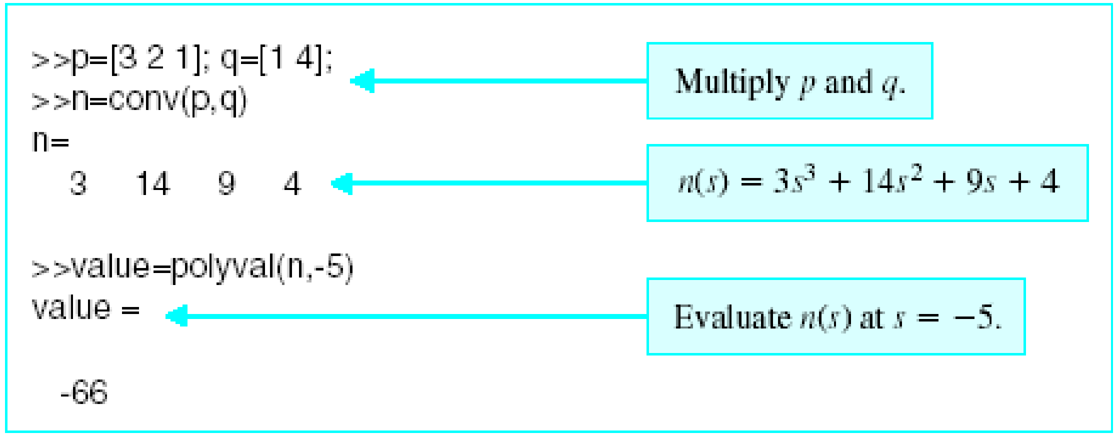


The Simulation of Systems Using MATLAB



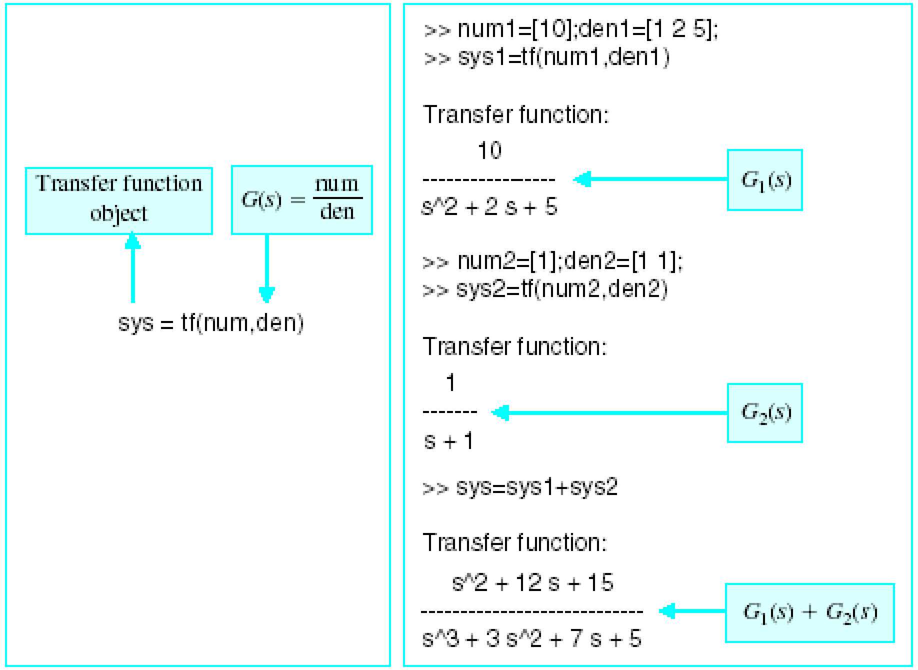
Entering the polynomial p(s) = s3 1 3s2 1 4 and calculating its roots.

The Simulation of Systems Using MATLAB



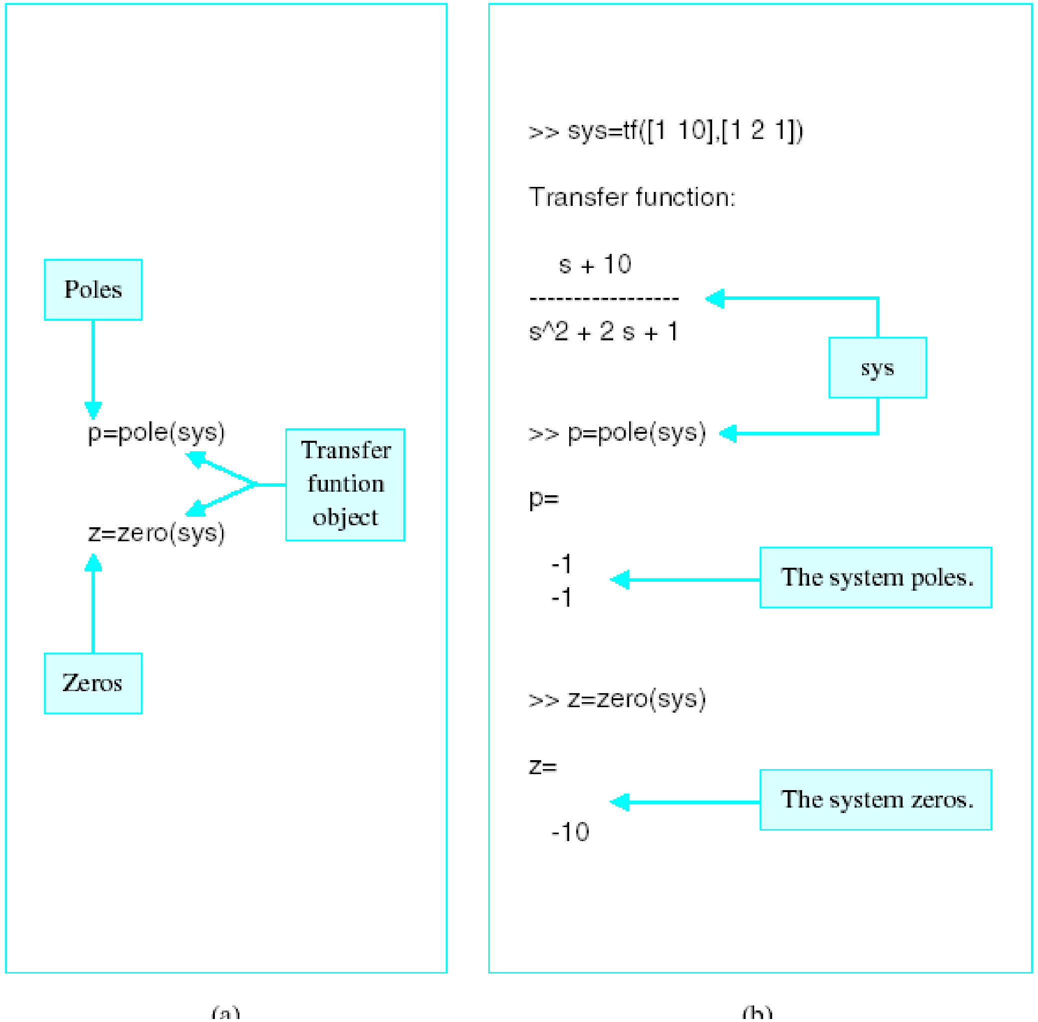
Using conv and polyval to multiply and evaluate the polynomials (3s2 1 2s 1 1)(s 1 4).

The Simulation of Systems Using MATLAB



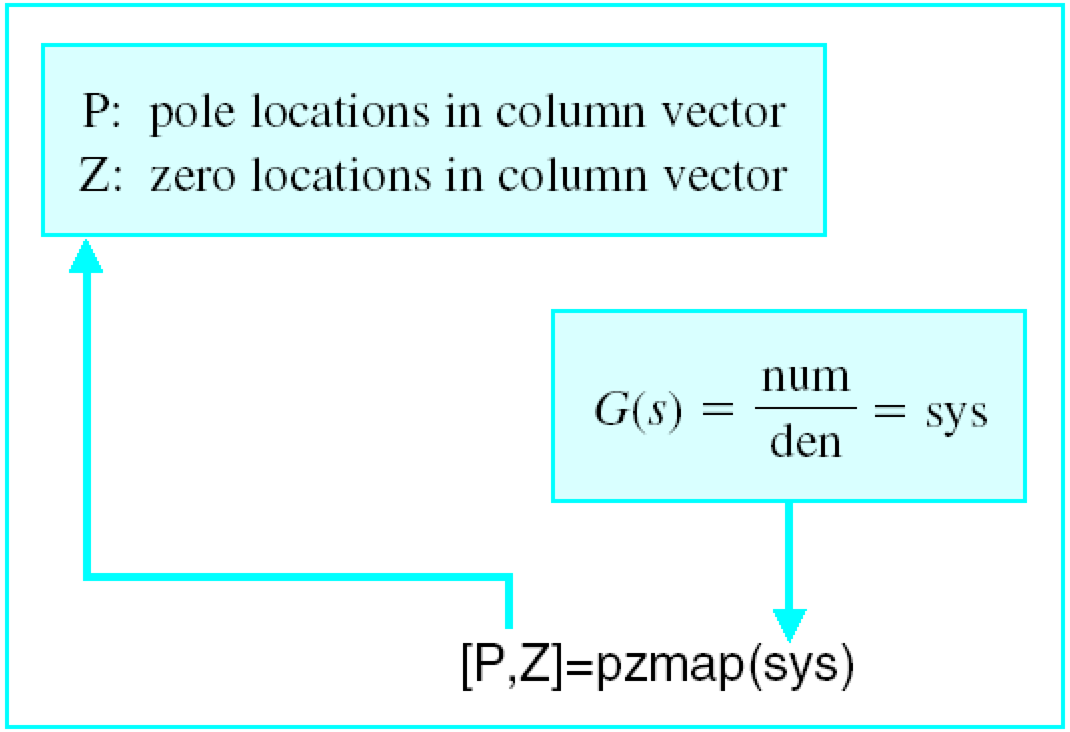
(a) The tf function.
(b) Using the tf function to create transfer function objects and adding them using the "1" operator.

The Simulation of Systems Using MATLAB



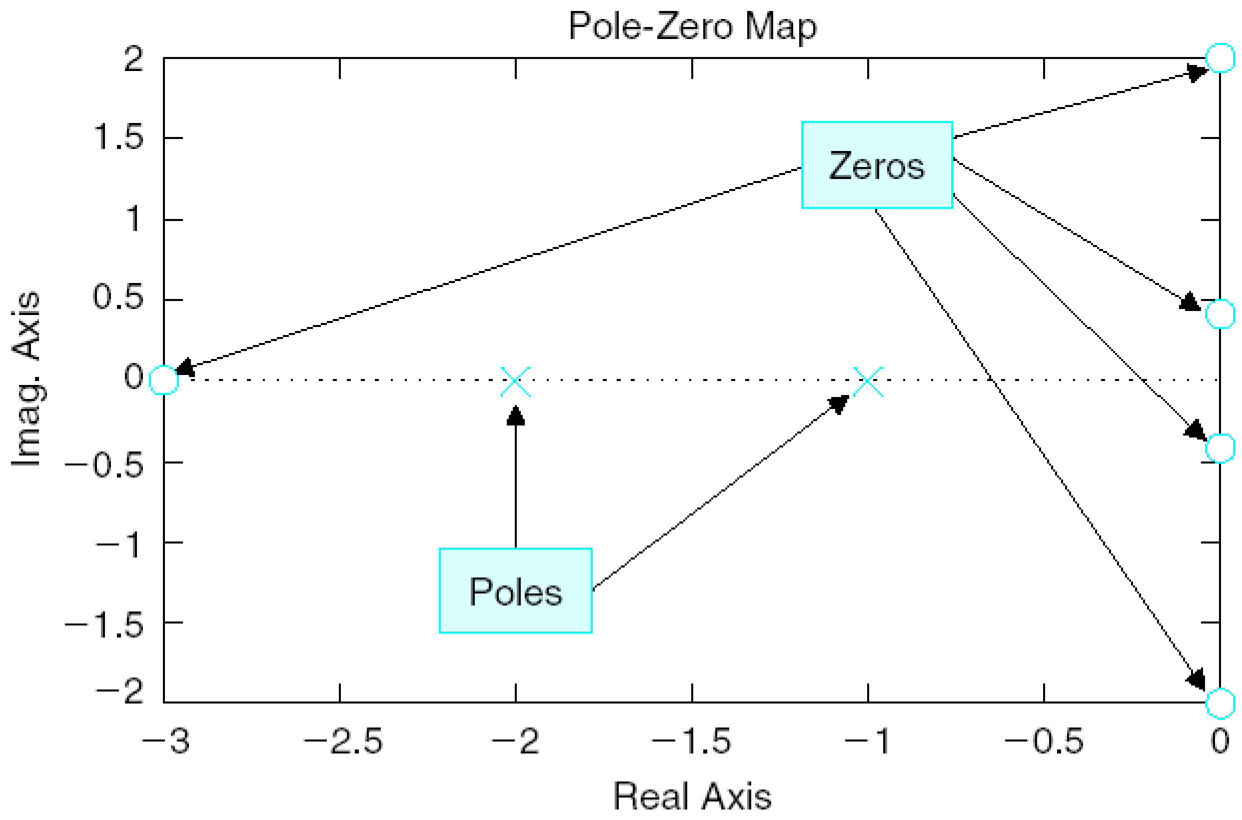
(a) The pole and zero functions.
(b) Using the pole and zero functions to compute the pole and zero locations of a linear system.

The Simulation of Systems Using MATLAB



The pzmap function.

The Simulation of Systems Using MATLAB



Pole–zero map for G(s)/H(s).

The Simulation of Systems Using MATLAB

```
>>numg=[6 0 1]; deng=[1 3 3 1];sysg=tf(numg,deng);
>>z=zero(sysg)

z =
     0 + 0.4082i
     0 - 0.4082i

>>p=pole(sysg)

p =
     -1.0000
     -1.0000 + 0.0000i
     -1.0000 -  0.0000i

>>n1=[1 1]; n2=[1 2]; d1=[1 2*i]; d2=[1 -2*i]; d3=[1 3];
>>numh=conv(n1,n2); denh=conv(d1,conv(d2,d3));
>>sysh=tf(numh/denh)

Transfer function:
      s^2 + 3 s + 2
-----------------------
s^3 + 3 s^2 + 4 s + 12

>>sys=sysg/sysh

Transfer function:
6 s^5 + 18 s^4 + 25 s^3 + 75 s^2 + 4 s +12
-------------------------------------------
   s^5 + 6 s^4 + 14 s^3 + 16 s^2 + 9 s + 2

>>pzmap(sys)
```

Compute poles and zeros of $G(s)$.

Expand $H(s)$.

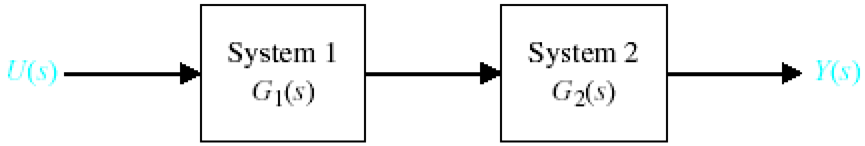
$H(s)$

$\frac{G(s)}{H(s)} = \text{sys}$

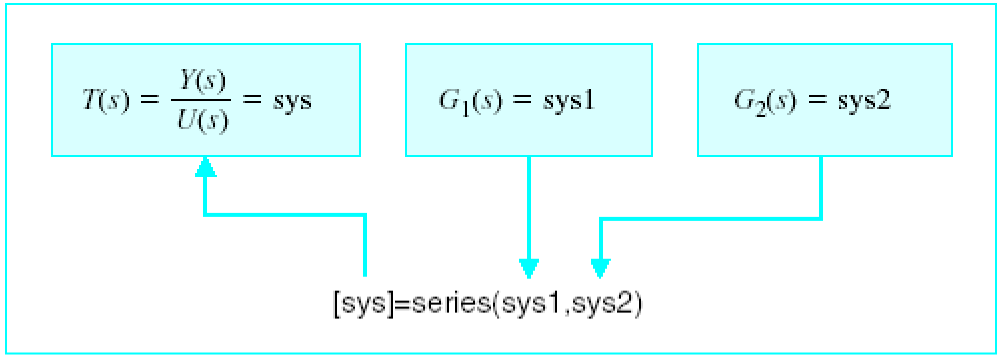
Pole-zero map.

Transfer function example for G(s) and H(s).

The Simulation of Systems Using MATLAB



(a)



(b)

(a) Block diagram. (b) The series function.

The Simulation of Systems Using MATLAB

(a)

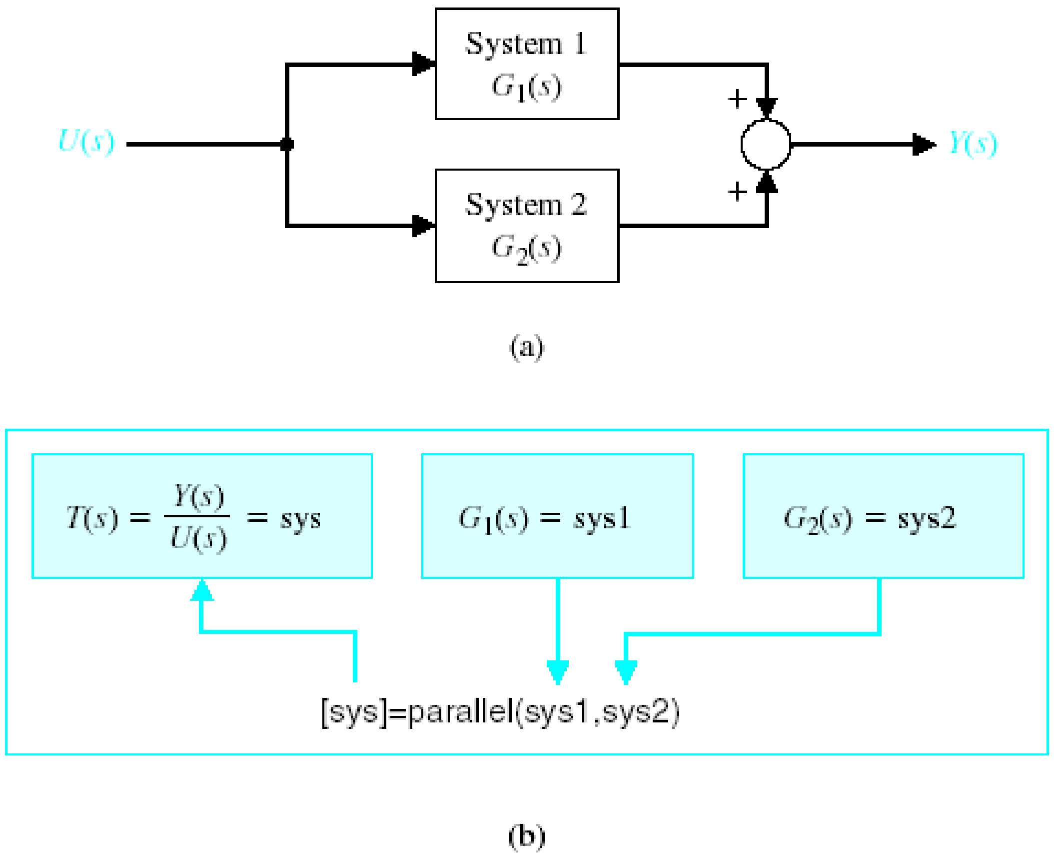
```
>>numg=[1]; deng=[500 0 0]; sysg=tf(numg,deng);
>>numh=[1 1]; denh=[1 2]; sysh=tf(numh,denh);
>>sys=series(sysg,sysh);
>>sys

Transfer function:
        s + 1
------------------
500 s^3 + 1000 s^2
```

← $G_c(s)G(s)$

(b)

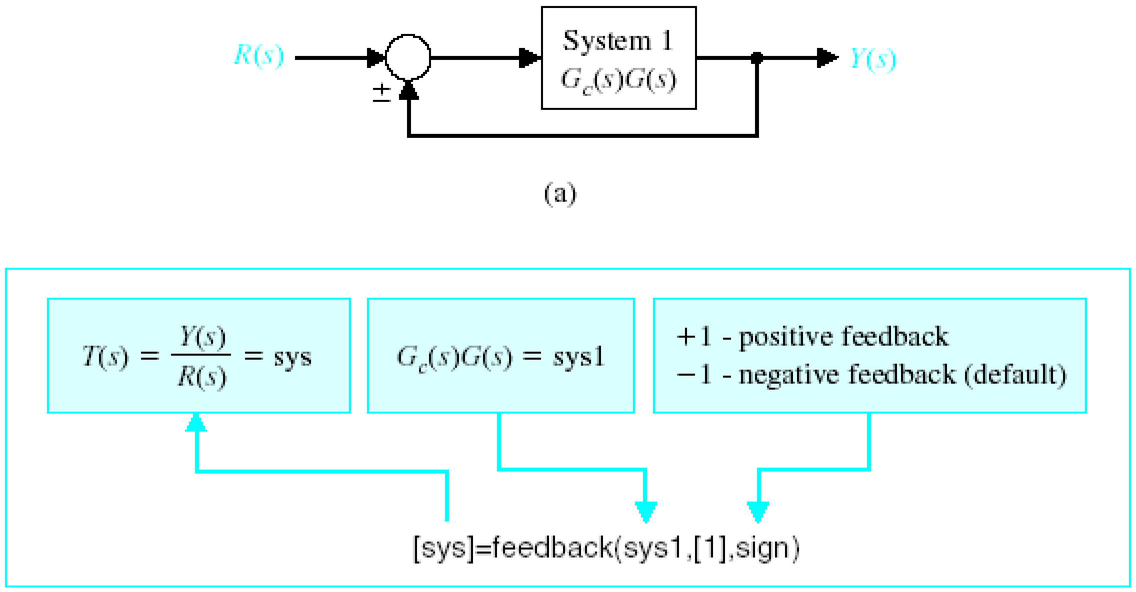
Application of the series function.

The Simulation of Systems Using MATLAB



(a) Block diagram.
(b) The parallel function.

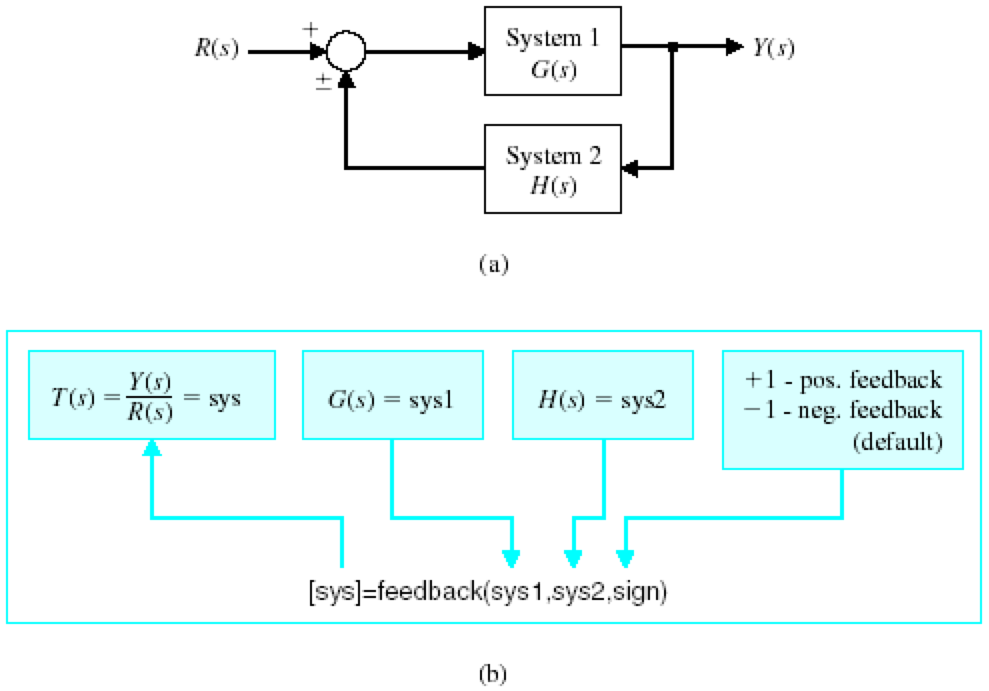
The Simulation of Systems Using MATLAB



(b)

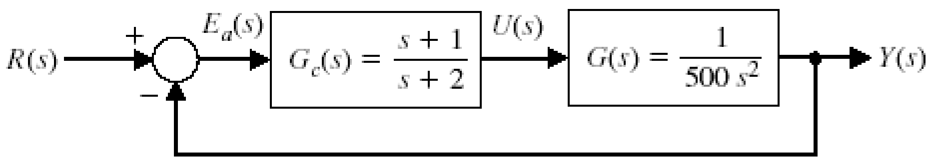
(a) Block diagram.
(b) The feedback function with unity feedback.

The Simulation of Systems Using MATLAB



(a) Block diagram.
(b) The feedback function.

The Simulation of Systems Using MATLAB



(a)

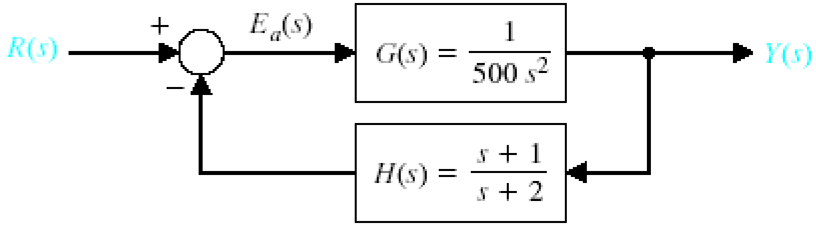
```
>>numg=[1]; deng=[500 0 0]; sys1=tf(numg,deng);
>>numc=[1 1]; denc=[1 2]; sys2=tf(numc,denc);
>>sys3=series(sys1,sys2);
>>sys=feedback(sys3,[1])

Transfer function:
          s + 1
-------------------------
500 s^3 + 1000 s^2 + s + 1
```

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}$$

(b)

(a) Block diagram.
(b) Application of the feedback function.

The Simulation of Systems Using MATLAB



(a)

```
>>numg=[1]; deng=[500 0 0]; sys1=tf(numg,deng);
>>numh=[1 1]; denh=[1 2]; sys2=tf(numh,denh);
>>sys=feedback(sys1,sys2);
>>sys

Transfer function:
          s + 2
-----------------------------
500 s^3 + 1000 s^2 + s + 1
```

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

(b)

Application of the feedback function:
(a) block diagram, (b) MATLAB script.

The Simulation of Systems Using MATLAB

```
>>ng1=[1]; dg1=[1 10]; sysg1=tf(ng1,dg1);
>>ng2=[1]; dg2=[1 1]; sysg2=tf(ng2,dg2);
>>ng3=[1 0 1]; dg3=[1 4 4]; sysg3=tf(ng3,dg3);
>>ng4=[1 1]; dg4=[1 6]; sysg4=tf(ng4,dg4);
>>nh1=[1 1]; dh1=[1 2]; sysh1=tf(nh1,dh1);
>>nh2=[2]; dh2=[1]; sysh2=tf(nh2,dh2);
>>nh3=[1]; dh3=[1]; sysh3=tf(nh3,dh3);
>>sys1=sys2/sys4;
>>sys2=series(sysg3,sysg4);
>>sys3=feedback(sys2,sysh1,+1);
>>sys4=series(sysg2,sys3);
>>sys5=feedback(sys4,sys1);
>>sys6=series(sysg1,sys5);
>>sys=feedback(sys6,[1]);

Transfer function:
          s^5 + 4 s^4 + 6 s^3 + 6 s^2 + 5 s + 2
-----------------------------------------------------------------
12 s^6 + 205 s^5 + 1066 s^4 + 2517 s^3 + 3128 s^2 + 2196 s + 712
```

Step 1

Step 2

Step 3

Step 4

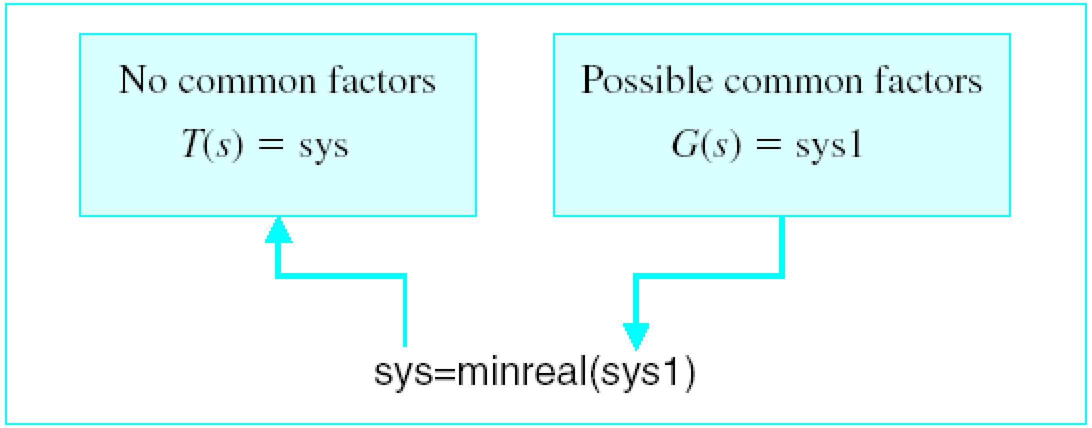
Step 5

error

Multiple-loop block reduction.

Sys1 = sysh2 / sysg4

The Simulation of Systems Using MATLAB



The minreal function.

```
>>num=[1 4 6 6 5 2]; den=[12 205 1066 2517 3128 2196 712];
>>sys1=tf(num,den);
>>sys=minreal(sys1);        <-- Cancel common factors.

Transfer function:

  0.08333 s^4 + 0.25 s^3 + 0.25 s^2 + 0.25 s + 0.1667
-------------------------------------------------------
 s^5 + 16.08 s^4 + 72.75 s^3 + 137 s^2 + 123.7 s + 59.33
```

Application of the minreal function.

The Simulation of Systems Using MATLAB

error

```
>>num1=[10]; den1=[1 1]; sys1=tf(num1,den1);
>>num2=[1]; den2=[2 0.5]; sys2=tf(num2,den2);
>>num3=[540]; den3=[1]; sys3=tf(num3,den3);
>>num3=[0.1]; den4=[1]; sys4=tf(num4,den4);
>>sys5=series(sys1,sys2);
>>sys6=feedback(sys5,sys4);
>>sys7=series(sys3,sys6);
>>sys=feedback(sys7,[1])
```

Eliminate inner loop.

Compute closed-loop transfer function.

Transfer function:

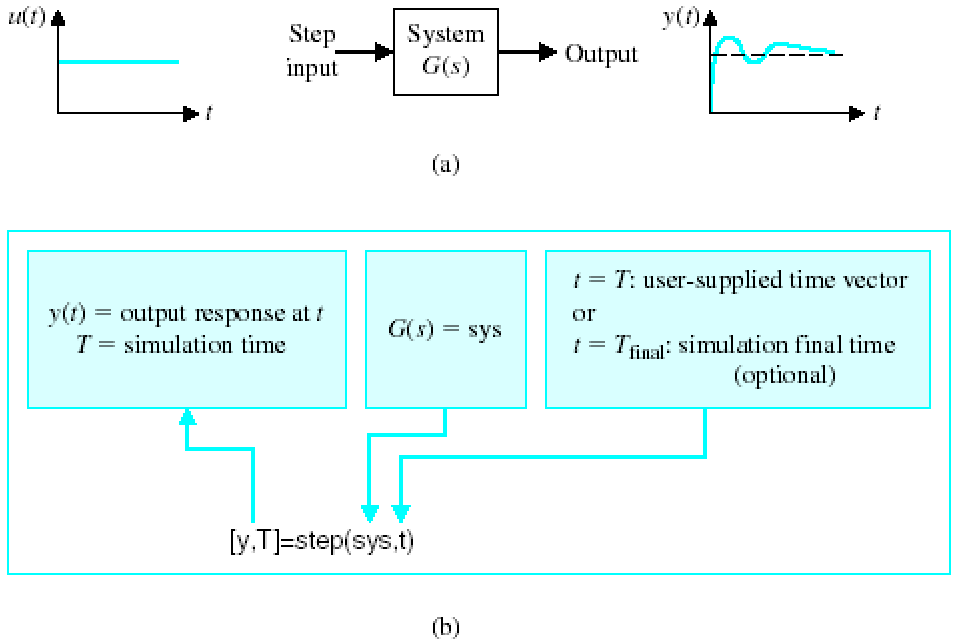
$$\frac{5400}{2\ s^2 + 2.5\ s + 5402}$$

$$\frac{\omega(s)}{\omega_d(s)}$$

Electric traction motor block reduction.

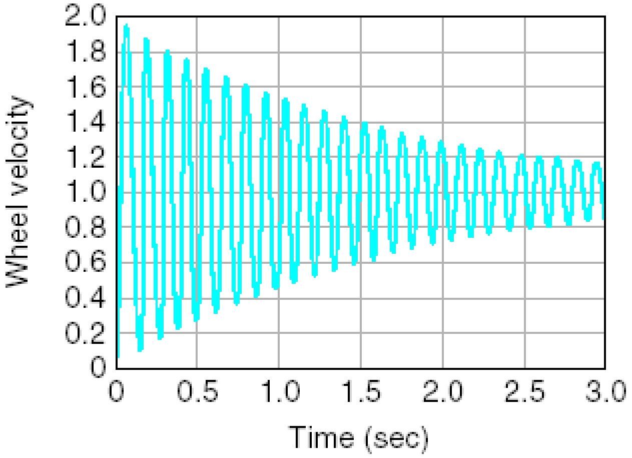
Num4=[0.1];

The Simulation of Systems Using MATLAB



The step function.

The Simulation of Systems Using MATLAB



(a)

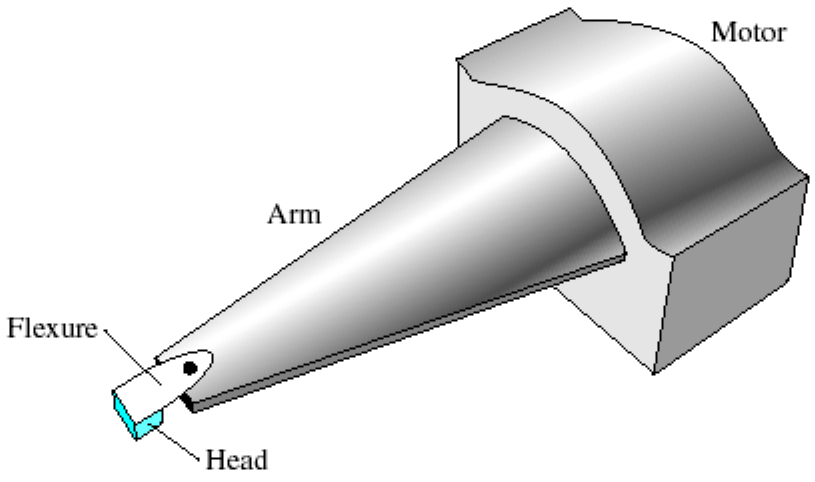
mresp.m

```
% This script computes the step
% response of the Traction Motor
% Wheel Velocity
%
num=[5400]; den=[2 2.5 5402]; sys=tf(num,den);
t=[0:0.005:3];
[y,t]=step(sys,t);
plot(t,y),grid
xlabel('Time (sec)')
ylabel('Wheel velocity')
```

(b)

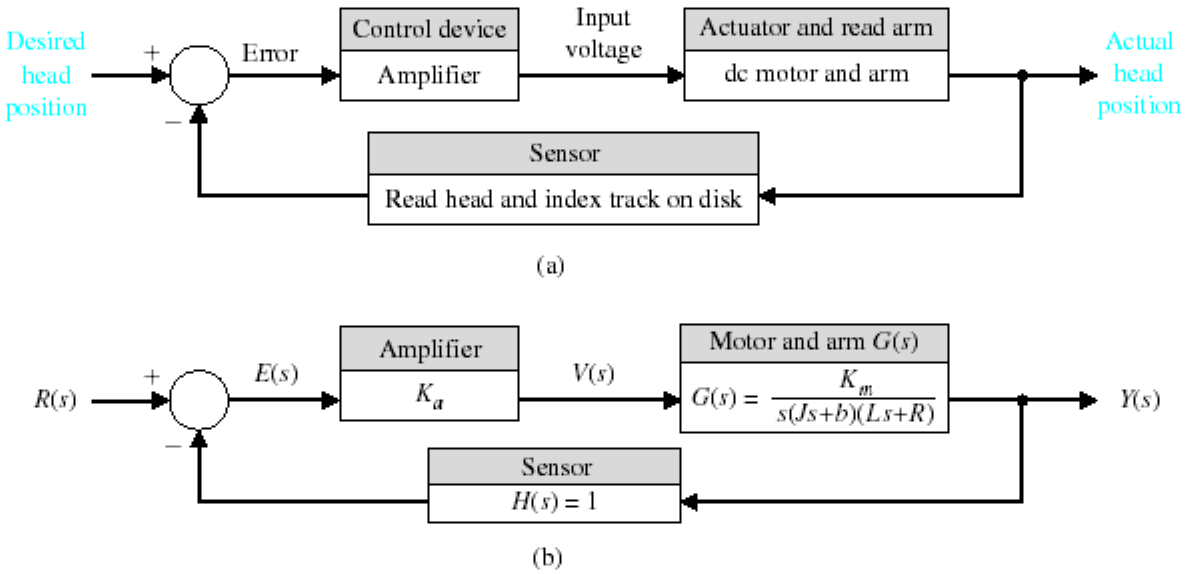
(a) Traction motor wheel velocity step response.
(b) Matlab script.

Sequential Design Example: Disk Drive Read System



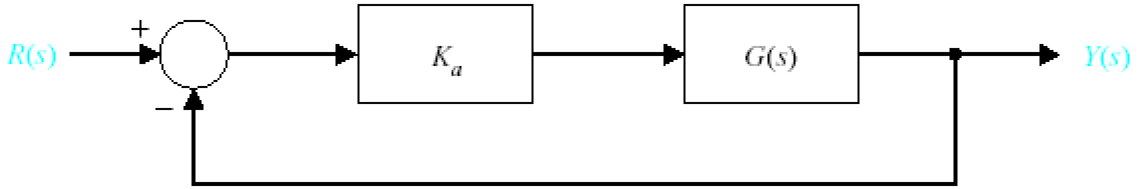
Head mount for reader, showing flexure.

Sequential Design Example: Disk Drive Read System

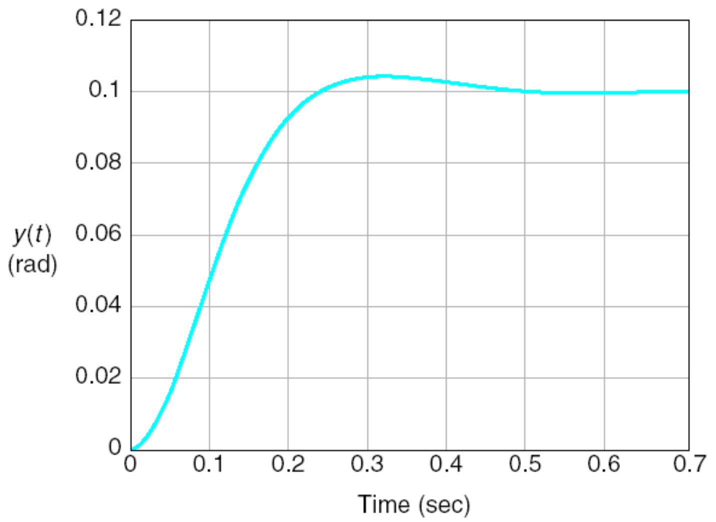


Block diagram model of disk drive read system.

Sequential Design Example: Disk Drive Read System

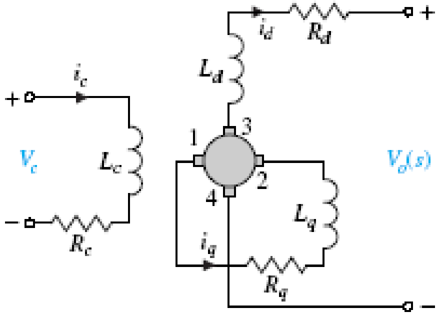


Block diagram of closed-loop system.

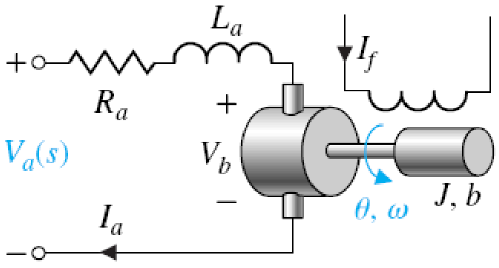


The system response of the system shown in Fig. 2.67 for $R(s) = \frac{0.1}{s}$

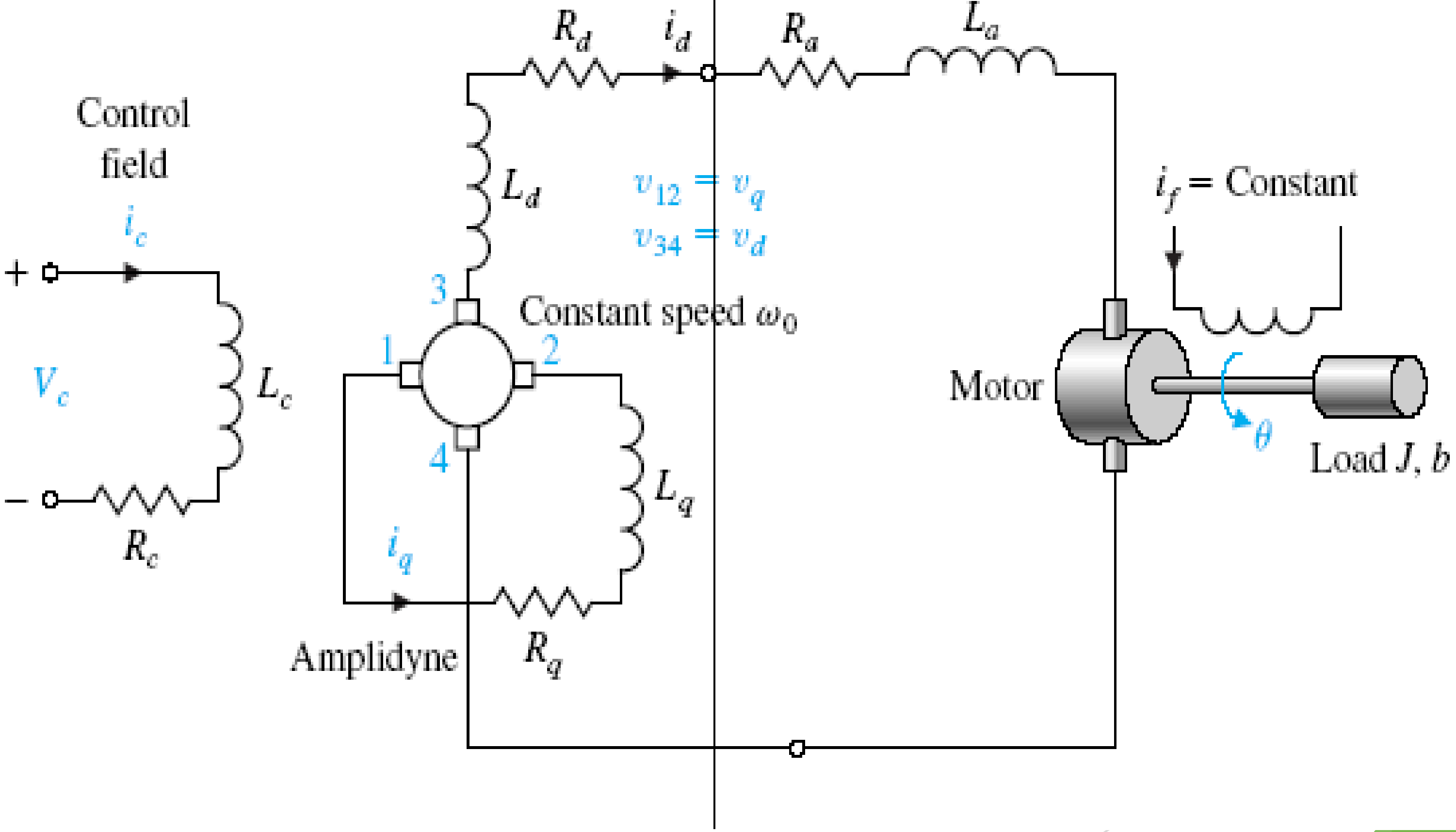
P2.11



i_c
V_c
L_c
R_c
i_d
R_d
L_d
$V_o(s)$
L_q
i_q
R_q

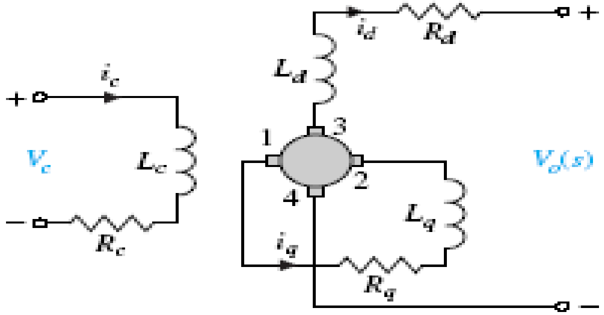


L_a
I_f
R_a
$V_a(s)$
V_b
I_a
θ, ω
J, b

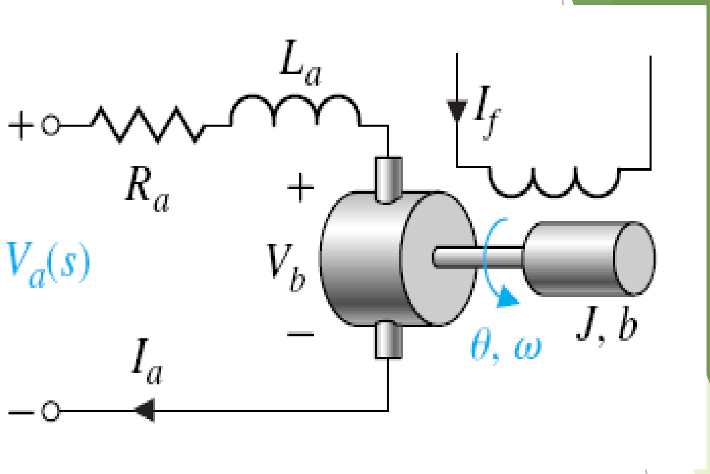


Control field
i_c
V_c
L_c
R_c
R_d
i_d
L_d
R_a
L_a
$v_{12} = v_q$
$v_{34} = v_d$
Constant speed ω_0
i_f = Constant
Motor
θ
Load J, b
L_q
i_q
Amplidyne
R_q

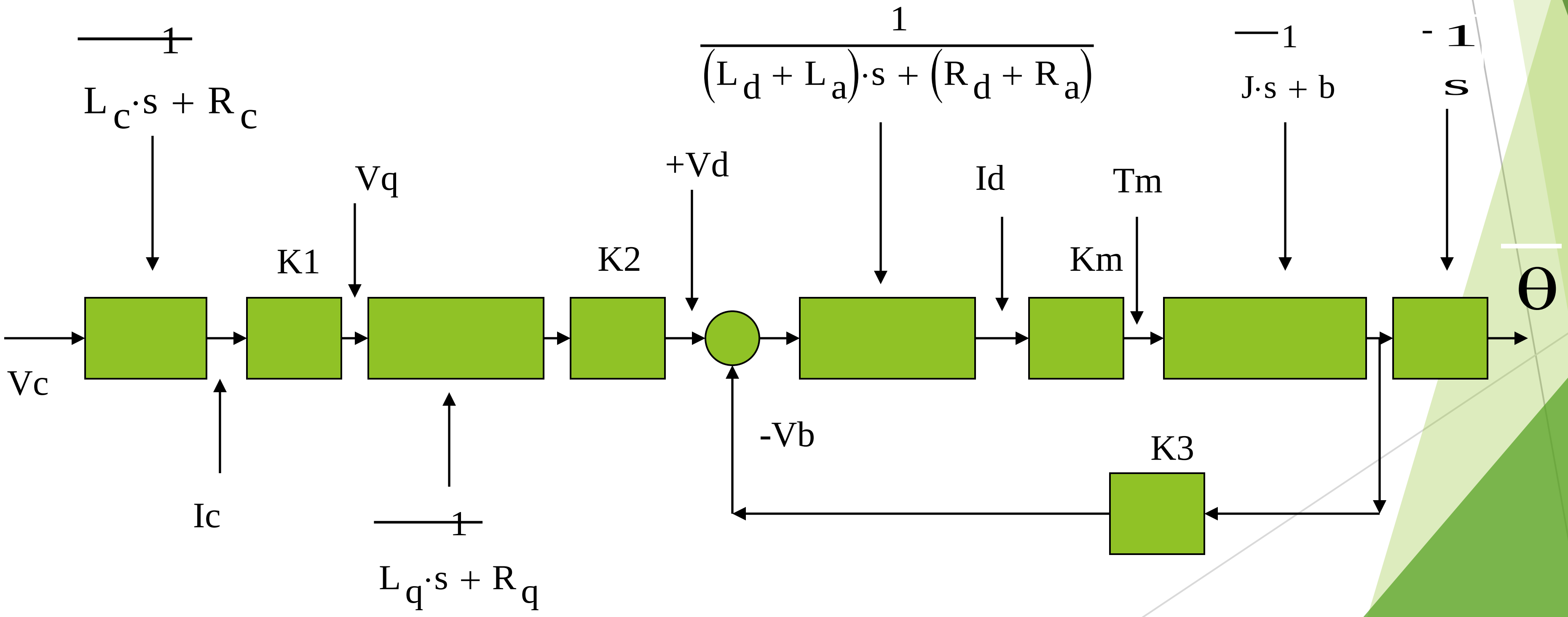
P2.11



+
i_c
V_c
L_c
−
R_c
i_d
R_d
L_d
1
2
3
4
L_q
i_q
R_q
$V_o(s)$
+
−

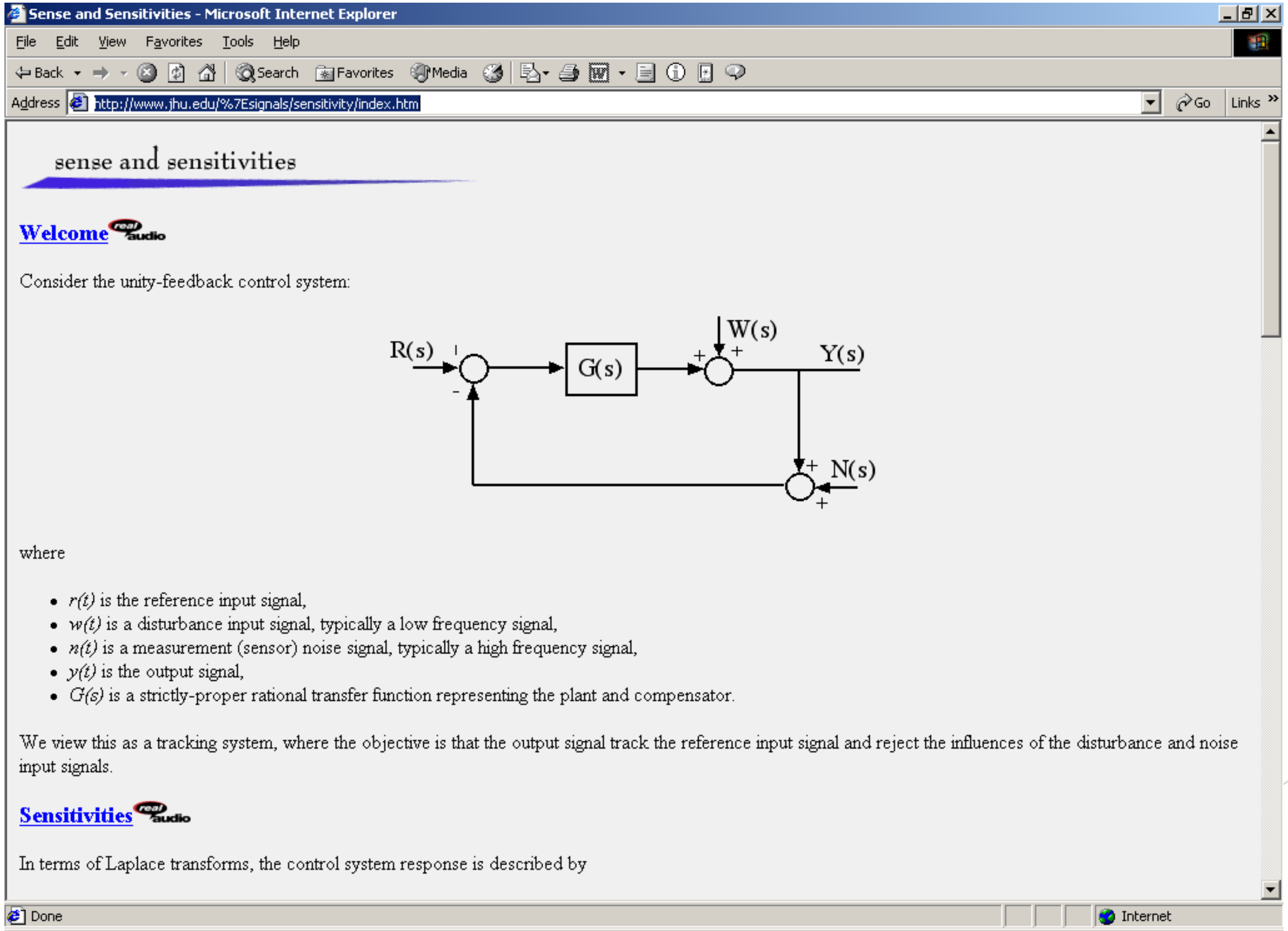


L_a
I_f
+
R_a
+
$V_a(s)$
V_b
−
θ, ω
J, b
I_a
−



$\frac{1}{L_c \cdot s + R_c}$
Vq
K1
+Vd
K2
$\frac{1}{(L_d + L_a) \cdot s + (R_d + R_a)}$
Id
Km
Tm
$\frac{1}{J \cdot s + b}$
$\frac{1}{s}$
θ
Vc
Ic
$\frac{1}{L_q \cdot s + R_q}$
-Vb
K3

http://www.jhu.edu/%7Esignals/sensitivity/index.htm



sense and sensitivities

Welcome

Consider the unity-feedback control system:

where

- *r(t)* is the reference input signal,
- *w(t)* is a disturbance input signal, typically a low frequency signal,
- *n(t)* is a measurement (sensor) noise signal, typically a high frequency signal,
- *y(t)* is the output signal,
- *G(s)* is a strictly-proper rational transfer function representing the plant and compensator.

We view this as a tracking system, where the objective is that the output signal track the reference input signal and reject the influences of the disturbance and noise input signals.

Sensitivities

In terms of Laplace transforms, the control system response is described by